Information technology – lecture 10

Number systems and representations. Computer arithmetic.

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Positional notation

Let:
$\beta \in N, \beta >= 2$ – the base
x_k – digits, $0 <= x_k < \beta$ with $k = -m, \ldots, n$

Notation:

$$x_\beta = (-1)^s [x_n x_{n-1} \ldots x_1 x_0 \,.\, x_{-1} x_{-2} \ldots x_{-m}] \quad x_n \neq 0$$

Interpretation:

$$x_\beta = (-1)^s \left(\sum_{k=-m}^{k=n} x_k \beta^k \right)$$

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Most common positional systems

decimal – base $\beta = 10$,
digits: 0,1,2,3,4,5,6,7,8,9

binary – base $\beta = 2$,
digits: 0,1

octal – base $\beta = 8$,
digits: 0,1,2,3,4,5,6,7

hexadecimal – base $\beta = 16$,
digits: 0,1,2,3,4,5,6,7,8,9,A,B,C,D,E,F

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Numbers in decimal system

$$175_{(10)} \rightarrow \text{digits} \rightarrow \; 1 \mid 7 \mid 5$$

$$175_{(10)} = (-1)^0 \left([\mathbf{1} \times 10^2] + [\mathbf{7} \times 10^1] + [\mathbf{5} \times 10^0]\right) \qquad (1)$$

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Numbers in binary system

$$175_{(10)} \rightarrow \text{digits} \rightarrow \; 1 \mid 0 \mid 1 \mid 0 \mid 1 \mid 1 \mid 1 \mid 1$$

$$175_{(10)} = (-1)^0 \Big([\mathbf{1} \times 2^7] + [\mathbf{0} \times 2^6] + [\mathbf{1} \times 2^5] + [\mathbf{0} \times 2^4] + [\mathbf{1} \times 2^3] + [\mathbf{1} \times 2^2] + [\mathbf{1} \times 2^1] + [\mathbf{1} \times 2^0] \Big)$$

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Numbers in octal system

$$175_{(10)} \rightarrow \text{digits} \rightarrow \;\; 2 \mid 5 \mid 7$$

$$175_{(10)} = (-1)^0 \left([\mathbf{2} \times 8^2] + [\mathbf{5} \times 8^1] + [\mathbf{5} \times 8^0] \right)$$

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Numbers in hexadecimal system

$$175_{(10)} \rightarrow \text{digits} \rightarrow \quad \text{A} \mid \text{F}$$

$$175_{(10)} = (-1)^0 \left([\mathbf{A} \times 16^1] + [\mathbf{F} \times 10^0] \right)$$

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Conversion binary → octal → decimal

binary	10101111		
	10	101	111
	2	4+1	4+2+1
octal	257		
	$2 \times 8^2 + 5 \times 8^1 + 7 \times 8^0$		
decimal	175		

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Conversion decimal $\longrightarrow$ other bases

While the quotient is not 0

    Divide the decimal number by the new base.
    Make the reminder the next digit to the left in the answer.
    Replace the decimal number with the quotient.

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Conversion decimal $\longrightarrow$ binary

175	integer division by 2: quotient, reminder		
	87	1	rightmost digit
	83	1	
	21	1	
	10	1	
	5	0	
	2	1	
	1	0	
	0	1	leftmost digit
10101111			$\longleftarrow$ result

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Conversion decimal → hexadecimal

175	integer division by 16: quotient, reminder		
	10	F	rightmost digit
	0	A	leftmost digit
AF			← result

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Arithmetic in binary system

Addition

```
  84              01010100
+ 91            + 01011011
 ---             ---------
 175              10101111
```

Multiplication

```
  35              00100011
*  5            * 00000101
 ---             ---------
 175              00100011
                 00000000
              + 00100011
               ----------
                  10101111
```

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Fixed point representation of real numbers

Let us assume that a real number is represented with N memory positions such that 1 position is hold for sign, $N-k-1$ positions for integer digits, k positions for the digits after the point.

Notation:

$$x = (-1)^s \left[a_{N-2}a_{N-3} \ldots a_k \blacksquare a_{k-1} \ldots a_0\right]$$

Interpretation:

$$x = (-1)^s \beta^{-k} \sum_{j=0}^{N-2} a_j \beta^j$$

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Floating point representation of real numbers

Let us assume that a real number is represented with N memory positions such that 1 position is hold for sign, $N - k - 1$ positions for integer digits, k positions for the digits after the point.

Notation:

$$x = (-1)^s \, [0 \, . \, a_1 a_2 \ldots a_t] \, \beta^e$$

Interpretation:

$$x = (-1)^s \times m \times \beta^{e-t}$$

where: t – the number of allowed significant digits a_j,
m – integer number called mantisa
e – integer number called exponent

The number zero has a separate representation.

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Floating point number formats

On 32 bit machine:

Single precision

1	8 bits	32 bits
s	e	n

Double precision

1	11 bits	52 bits
s	e	n

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Floating point number sets

Let us define a set of floating point numbers:

$$\mathcal{F}(\beta, t, L, U) = \{0\} \cup \{x \in R : x = (1)^s \beta^e \sum_{i=1}^{t} a_i \beta^- i\}$$

where: t – number of significant digits
β – base,
$0 < a_i \leqslant \beta - 1$ – digits
range (L, U) such that $L \leqslant e \leqslant U$

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Normalisation of floating point number representation

To enforce uniqueness in number representation it is assumed that:

$$a_1 \neq 0$$
$$m \geqslant \beta^{t-1}$$

Such representation is called **normalized**.
With such setup a_1 is the primary significant digit, a_t last significant digit.

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Distribution of floating point numbers

Floating point numbers are not equally spaced along the real line

machine epsilon – the smallest number in the set of floating point numbers such that $1 + \varepsilon_m > 1$

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IEEE 754 Standard

By choosing different set of values for β, t, L and U it is possible to build multiple floating point number systems. Thus the necessity of defining a standard for floating point arithmetic. One of the most widely-used is IEEE Standard for Floating Point Arithmetics (IEEE 754). The standard defines:

- arithmetic formats (binary and decimal)
- interchange formats,
- rounding algorithms,
- operations (arithmetic and other)
- exception handling

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