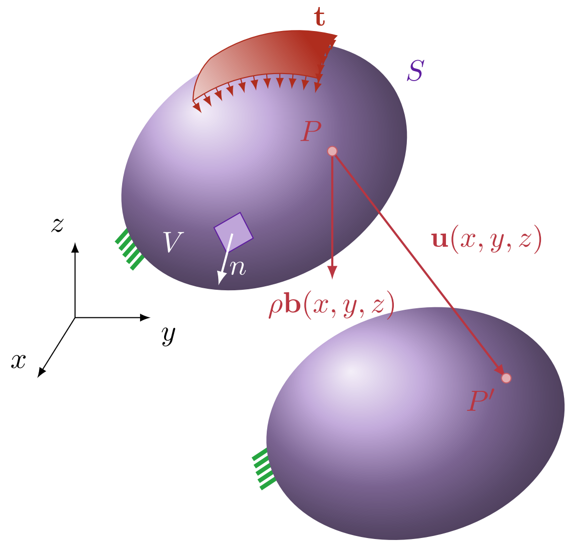
2. MES w mechanice ośrodka ciągłego

2.1. Stan równowagi



Wektor gęstości sił masowych [N/m^3]

$$\rho\mathbf{b} = \rho \begin{bmatrix} 0 \\ 0 \\ -g \end{bmatrix}$$

Wektor gęstości sił powierzchniowych [N/m^2]

$$\mathbf{t} = \begin{bmatrix} t_x \\ t_y \\ t_z \end{bmatrix}$$

Równanie równowagi ciała

$$\int_S \mathbf{t} \mathrm{d}S + \int_V \rho\mathbf{b} \mathrm{d}V = 0$$

Statyczne warunki brzegowe

$$\mathbf{t} = \boldsymbol{\sigma}\mathbf{n}$$

gdzie $\boldsymbol{\sigma}$ – tensor naprężeń

Wykorzystując twierdzenie Greena–Gaussa–Ostrogradzkiego

$$\int_S \boldsymbol{\sigma}\mathbf{n} \mathrm{d}S = \int_V \mathrm{div}\boldsymbol{\sigma} \mathrm{d}V$$

Równania Naviera

$$\int_V (\mathrm{div}\boldsymbol{\sigma} + \rho\mathbf{b})\,\mathrm{d}V = 0 \Longleftrightarrow \mathrm{div}\boldsymbol{\sigma} + \rho\mathbf{b} = 0 \quad \forall P \in V$$

$$\sigma_{ij,j} + \rho b_i = 0 + \text{w.b.}$$

Notacja tensorowa

$$t = \boldsymbol{\sigma}\mathbf{n} \qquad \text{gdzie } \boldsymbol{\sigma} = \begin{bmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{xy} & \sigma_y & \tau_{yz} \\ \tau_{xz} & \tau_{yz} & \sigma_z \end{bmatrix}, \quad \mathbf{n} = \begin{bmatrix} n_x \\ n_y \\ n_z \end{bmatrix}$$

Notacja Voigt'a

$$t = \mathbf{m}^{\mathrm{T}}\mathbf{s} \qquad \text{gdzie } \mathbf{s} = \begin{bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{yz} \\ \tau_{xz} \\ \tau_{xy} \end{bmatrix}, \quad \mathbf{m} = \begin{bmatrix} n_x & 0 & 0 \\ 0 & n_y & 0 \\ 0 & 0 & n_z \\ 0 & n_z & n_y \\ n_z & 0 & n_x \\ n_y & n_x & 0 \end{bmatrix}$$

$$\mathrm{div}\boldsymbol{\sigma} = \mathbf{L}^{\mathrm{T}}\mathbf{s} \quad \text{gdzie } \mathbf{L} = \begin{bmatrix} \frac{\partial}{\partial x} & 0 & 0 \\ 0 & \frac{\partial}{\partial y} & 0 \\ 0 & 0 & \frac{\partial}{\partial z} \\ 0 & \frac{\partial}{\partial z} & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} & 0 & \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} & 0 \end{bmatrix} \qquad \text{gdzie } \mathbf{L} \text{ – macierz operatorów różniczkowych}$$

Równanie Naviera w notacji Voigt'a

$$\mathbf{L}^{\mathrm{T}}\mathbf{s} + \rho\mathbf{b} = 0$$

Sformułowanie słabe – funkcja wagowa $w \cong \delta\mathbf{u}$ – kinematycznie dopuszczalna wariacja przemieszczenia (zgodna z kinematycznymi warunkami brzegowymi – zasada prac wirtualnych)

$$\int_V (\delta\mathbf{u})^{\mathrm{T}} \left(\mathbf{L}^{\mathrm{T}}\mathbf{s} + \rho\mathbf{b}\right) \mathrm{d}V = 0 \quad \forall\delta\mathbf{u}$$

$$-\int_V (\mathbf{L}\delta\mathbf{u})^{\mathrm{T}}\mathbf{s}\mathrm{d}V + \int_S (\delta\mathbf{u})^{\mathrm{T}} \overbrace{\mathbf{m}^{\mathrm{T}}\mathbf{s}}^{\mathbf{t}} \mathrm{d}S + \int_V (\delta\mathbf{u})^{\mathrm{T}}\rho\mathbf{b}\mathrm{d}V = 0$$

$$-\int_V (\mathbf{L}\delta\mathbf{u})^{\mathrm{T}}\mathbf{s}\mathrm{d}V + \int_S (\delta\mathbf{u})^{\mathrm{T}}\mathbf{t}\mathrm{d}S + \int_V (\delta\mathbf{u})^{\mathrm{T}}\rho\mathbf{b}\mathrm{d}V = 0$$

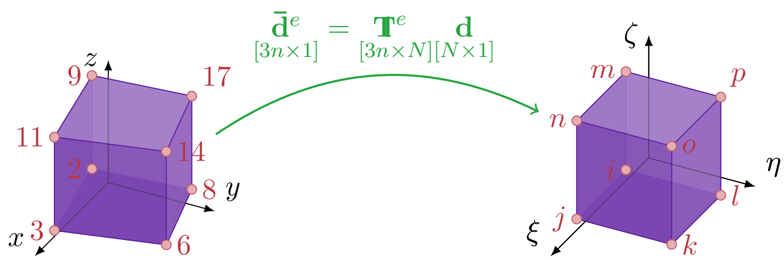
$$\underbrace{\int_V (\mathbf{L}\delta\mathbf{u})^{\mathrm{T}}\mathbf{s}\mathrm{d}V}_{\text{praca sił wewnętrznych}} = \underbrace{\int_S (\delta\mathbf{u})^{\mathrm{T}}\mathbf{t}\mathrm{d}S + \int_V (\delta\mathbf{u})^{\mathrm{T}}\rho\mathbf{b}\mathrm{d}V}_{\text{praca sił zewnętrznych}}$$

2.2. Dyskretyzacja MES (n=LWE, N=LSSU, E=LEU)

Aproksymacja pola przemieszczeń

$$\mathbf{u}^{eh} = \sum_{i=1}^{n} N_i^e(\xi,\eta,\zeta)\bar{\mathbf{d}}_i^e = \mathbf{N}^e\bar{\mathbf{d}}^e$$

$$\underset{[3\times 3n]}{\mathbf{N}^e} = \left[\begin{array}{ccc|c|ccc} N_1^e & 0 & 0 & \dots & N_n^e & 0 & 0 \\ 0 & N_1^e & 0 & \dots & 0 & N_n^e & 0 \\ 0 & 0 & N_1^e & \dots & 0 & 0 & N_n^e \end{array}\right] \qquad \underset{[3n\times 1]}{\bar{\mathbf{d}}^e} = \left[\begin{array}{c} \bar{\mathbf{d}}_1^e \\ \dots \\ \bar{\mathbf{d}}_n^e \end{array}\right]$$



$\mathbb{T}^e$ – macierz transformacji uwzględniająca topologię i cosinusy kierunkowe pomiędzy osiami układu globalnego i lokalnego

2.3. Równanie równowagi układu zdyskretyzowanego

Równanie równowagi ($\rho\mathbf{b}^e = \mathbf{f}^e$ – wektor sił objętościowych)

$$\sum_{e=1}^{E}\left\{\int_{V^e}(\mathbf{L}^e\delta\mathbf{u}^e)^{\mathrm{T}}\mathbf{s}^e\mathrm{d}V^e - \int_{S^e}(\delta\mathbf{u}^e)^{\mathrm{T}}\mathbf{t}^e\mathrm{d}S^e - \int_{V^e}(\delta\mathbf{u}^e)^{\mathrm{T}}\mathbf{f}^e\mathrm{d}V^e\right\} = 0$$

$$\sum_{e=1}^{E}\left\{\int_{V^e}(\overbrace{\mathbf{L}^e\mathbf{N}^e}^{\mathbf{B}^e}\delta\bar{\mathbf{d}}^e)^{\mathrm{T}}\mathbf{s}^e\mathrm{d}V^e - \int_{S^e}(\mathbf{N}^e\delta\bar{\mathbf{d}}^e)^{\mathrm{T}}\mathbf{t}^e\mathrm{d}S^e - \int_{V^e}(\mathbf{N}^e\delta\bar{\mathbf{d}}^e)^{\mathrm{T}}\mathbf{f}^e\mathrm{d}V^e\right\} = 0$$

$$\sum_{e=1}^{E}(\overbrace{\delta\bar{\mathbf{d}}^e}^{\mathbb{T}^e\delta\mathbf{d}})^{\mathrm{T}}\left\{\int_{V^e}\mathbf{B}^{e\mathrm{T}}\mathbf{s}^e\mathrm{d}V^e - \int_{S^e}\mathbf{N}^{e\mathrm{T}}\mathbf{t}^e\mathrm{d}S^e - \int_{V^e}\mathbf{N}^{e\mathrm{T}}\mathbf{f}^e\mathrm{d}V^e\right\} = 0$$

$$(\delta\mathbf{d})^{\mathrm{T}}\left\{\sum_{e=1}^{E}\mathbb{T}^{e\mathrm{T}}\left\{\int_{V^e}\mathbf{B}^{e\mathrm{T}}\mathbf{s}^e\mathrm{d}V^e - \int_{S^e}\mathbf{N}^{e\mathrm{T}}\mathbf{t}^e\mathrm{d}S^e - \int_{V^e}\mathbf{N}^{e\mathrm{T}}\mathbf{f}^e\mathrm{d}V^e\right\}\right\} = 0$$

$$\delta\mathbf{d} \neq \mathbf{0} \Longrightarrow \sum_{e=1}^{E}\mathbb{T}^{e\mathrm{T}}\left\{\int_{V^e}\mathbf{B}^{e\mathrm{T}}\mathbf{s}^e\mathrm{d}V^e - \int_{S^e}\mathbf{N}^{e\mathrm{T}}\mathbf{t}^e\mathrm{d}S^e - \int_{V^e}\mathbf{N}^{e\mathrm{T}}\mathbf{f}^e\mathrm{d}V^e\right\} = 0$$

$$\underbrace{\sum_{e=1}^{E}\mathbb{T}^{e\mathrm{T}}\left\{\int_{V^e}\mathbf{B}^{e\mathrm{T}}\mathbf{s}^e\mathrm{d}V^e\right\}}_{\text{siły wewnętrzne}} = \underbrace{\sum_{e=1}^{E}\mathbb{T}^{e\mathrm{T}}\left\{\int_{S^e}\mathbf{N}^{e\mathrm{T}}\mathbf{t}^e\mathrm{d}S^e + \int_{V^e}\mathbf{N}^{e\mathrm{T}}\mathbf{f}^e\mathrm{d}V^e\right\}}_{\text{siły zewnętrzne}}$$

Uwzględnienie związków kinematycznych i konstytutywnych

- liniowa sprężystość (zw. konstytutywny) : $\mathbf{s} = \mathbf{D}\mathbf{e}$
- liniowy związek kinematyczny : $\mathbf{e} = \mathbf{L}\mathbf{u}$

$$\mathbf{s}^e = \mathbf{D}^e\mathbf{L}^e\mathbf{u}^e = \mathbf{D}^e\mathbf{L}^e\mathbf{N}^e\bar{\mathbf{d}}^e = \mathbf{D}^e\mathbf{B}^e\mathbb{T}^e\mathbf{d}$$

$$\sum_{e=1}^{E}\mathbb{T}^{e\mathrm{T}}\left\{\int_{V^e}\mathbf{B}^{e\mathrm{T}}\mathbf{D}^e\mathbf{B}^e\mathrm{d}V^e\right\}\mathbb{T}^e\mathbf{d} = \sum_{e=1}^{E}\mathbb{T}^{e\mathrm{T}}\left\{\int_{S^e}\mathbf{N}^{e\mathrm{T}}\mathbf{t}^e\mathrm{d}S^e + \int_{V^e}\mathbf{N}^{e\mathrm{T}}\mathbf{f}^e\mathrm{d}V^e\right\}$$

$$\sum_{e=1}^{E}\mathbb{T}^{e\mathrm{T}}\left\{\underbrace{\int_{V^e}\mathbf{B}^{e\mathrm{T}}\mathbf{D}^e\mathbf{B}^e\mathrm{d}V^e}_{\bar{\mathbf{K}}^e}\right\}\mathbb{T}^e\mathbf{d} = \sum_{e=1}^{E}\mathbb{T}^{e\mathrm{T}}\left\{\underbrace{\int_{S^e}\mathbf{N}^{e\mathrm{T}}\mathbf{t}^e\mathrm{d}S^e}_{\bar{\mathbf{p}}_{\mathrm{b}}^e} + \underbrace{\int_{V^e}\mathbf{N}^{e\mathrm{T}}\mathbf{f}^e\mathrm{d}V^e}_{\bar{\mathbf{p}}^e}\right\}$$

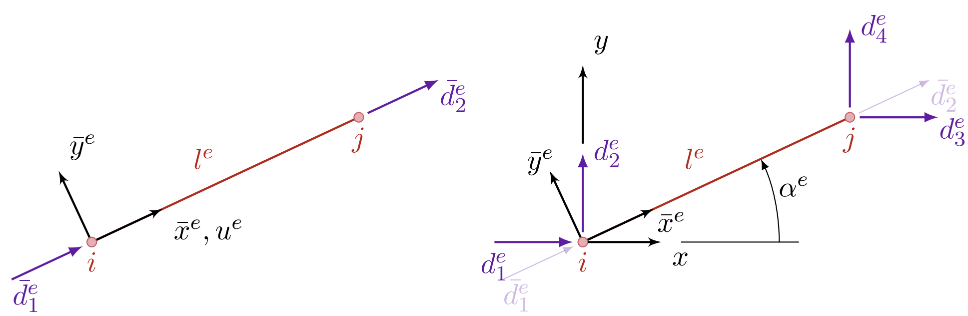
$$\sum_{e=1}^{E}\mathbb{T}^{e\mathrm{T}}\bar{\mathbf{K}}^e\mathbb{T}^e\mathbf{d} = \sum_{e=1}^{E}\mathbb{T}^{e\mathrm{T}}\bar{\mathbf{p}}_{\mathrm{b}}^e + \sum_{e=1}^{E}\mathbb{T}^{e\mathrm{T}}\bar{\mathbf{p}}^e$$

$$\underbrace{\sum_{e=1}^{E}\mathbb{T}^{e\mathrm{T}}\bar{\mathbf{K}}^e\mathbb{T}^e}_{\mathbf{K}}\mathbf{d} = \underbrace{\sum_{e=1}^{E}\mathbb{T}^{e\mathrm{T}}\bar{\mathbf{p}}_{\mathrm{b}}^e}_{\mathbf{p}_{\mathrm{b}}} + \underbrace{\sum_{e=1}^{E}\mathbb{T}^{e\mathrm{T}}\bar{\mathbf{p}}^e}_{\mathbf{p}}$$

$$\mathbf{K}\mathbf{d} = \mathbf{p}_{\mathrm{b}} + \mathbf{p}$$

2.4. Elementy prętowe

2.4.1. Element kratowy



Wektor funkcji przemieszczeń

$$\mathbf{u} = \{u(x)\}$$

Wektor uogólnionych odkształceń

$$\mathbf{e} = \{\varepsilon_x\}$$

Wektor uogólnionych naprężeń

$$\mathbf{s} = \{N(x)\}$$

Wektor obciążenia po długości elementu

$$\mathbf{f} = \{f_x\}$$

Macierz związków konstytutywnych

$$\mathbf{D} = \begin{bmatrix} EA \end{bmatrix}$$

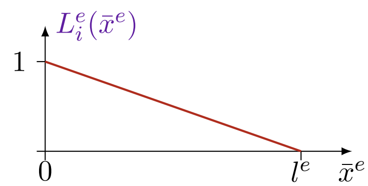
Macierz operatorów różniczkowych

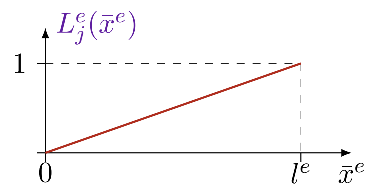
$$\mathbf{L} = \begin{bmatrix} \dfrac{\mathrm{d}}{\mathrm{d}x} \end{bmatrix}$$

Aproksymacja

$$\mathbf{u}^e(x) = \mathbf{N}^e(x)\bar{\mathbf{d}}^e$$

$$\mathbf{N}^e = \begin{bmatrix} L_i^e & L_j^e \end{bmatrix}, \bar{\mathbf{d}}^e = \begin{bmatrix} \bar{d}_1 \\ \bar{d}_2 \end{bmatrix}$$





Macierz sztywności

$$\bar{\mathbf{K}}^e = \int_0^{l^e} \mathbf{B}^{e\mathrm{T}} \mathbf{D}^e \mathbf{B}^e \mathrm{d}\bar{x}^e$$

$$\bar{\mathbf{K}}^e = \begin{bmatrix} \frac{EA}{l} & -\frac{EA}{l} \\ -\frac{EA}{l} & \frac{EA}{l} \end{bmatrix}^e$$

Macierz transformacji
$c = \cos(\alpha^e)$ **i** $s = \sin(\alpha^e)$

$$\mathbf{T}^e = \begin{bmatrix} c & s & 0 & 0 \\ 0 & 0 & c & s \end{bmatrix}$$

Wzory transformacyjne

$$\mathbf{K}^e = \mathbf{T}^{e\mathrm{T}} \bar{\mathbf{K}}^e \mathbf{T}^e$$

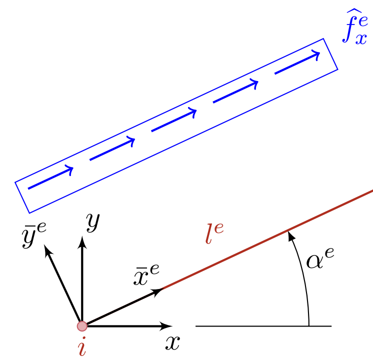
$$\mathbf{x}^e = \mathbf{T}^{e\mathrm{T}} \bar{\mathbf{x}}^e, \qquad \bar{\mathbf{x}}^e = \mathbf{T}^e \mathbf{x}^e$$

Wektor zastępników

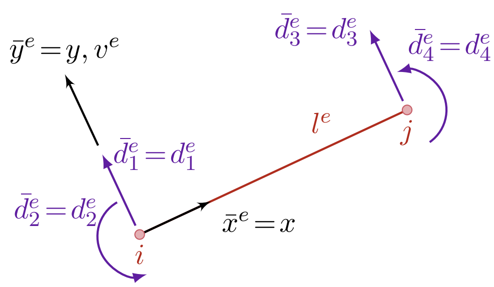
$$\bar{\mathbf{p}}^e = \int_0^{l^e} \mathbf{N}^{e\mathrm{T}} f^e \mathrm{d}\bar{x}^e$$

dla $f_x = \mathrm{const} = \widehat{f}_x$

$$\bar{\mathbf{p}}^e = \left\{ \frac{\widehat{f}_x l}{2},\ \frac{\widehat{f}_x l}{2} \right\}^e$$



2.4.2. Element belkowy



Wektor funkcji przemieszczeń

$$\mathbf{u} = \{v(x)\}$$

Wektor uogólnionych odkształceń

$$\mathbf{e} = \{\kappa\}$$

Wektor uogólnionych naprężeń

$$\mathbf{s} = \{M(x)\}$$

Wektor obciążenia po długości elementu

$$\mathbf{f} = \{f_y\}$$

Macierz związków konstytutywnych

$$\mathbf{D} = \left[\ EI\ \right]$$

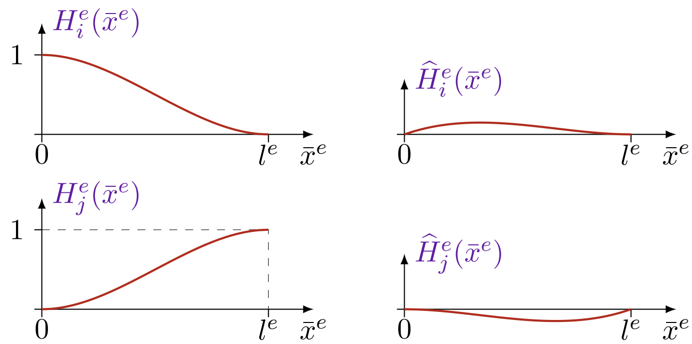
Macierz operatorów różniczkowych

$$\mathbf{L} = \left[\ -\frac{\mathrm{d}^2}{\mathrm{d}x^2}\ \right]$$

Aproksymacja

$$\mathbf{u}^e(x) = \mathbf{N}^e(x)\bar{\mathbf{d}}^e$$

$$\mathbf{N}^e = \begin{bmatrix} H_i^e & \widehat{H}_i^e & H_j^e & \widehat{H}_j^e \end{bmatrix}, \bar{\mathbf{d}}^e = \begin{bmatrix} \bar{d}_1 \\ \bar{d}_2 \\ \bar{d}_3 \\ \bar{d}_4 \end{bmatrix}$$



Macierz sztywności

$$\bar{\mathbf{K}}^e = \int_0^{l^e} \mathbf{B}^{e\mathrm{T}}\mathbf{D}^e\mathbf{B}^e \mathrm{d}\bar{x}^e$$

$$\bar{\mathbf{K}}^e = \frac{EI}{l^3}\begin{bmatrix} 12 & 6l & -12 & 6l \\ 6l & 4l^2 & -6l & 2l^2 \\ -12 & -6l & 12 & -6l \\ 6l & 2l^2 & -6l & 4l^2 \end{bmatrix}^e$$

Macierz transformacji

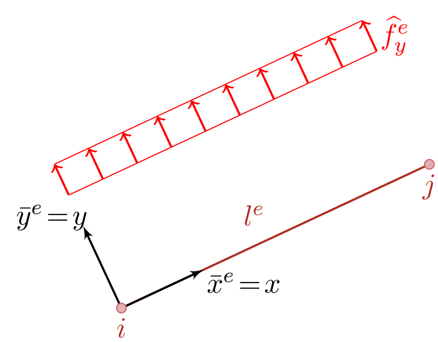
brak

Wektor zastępników

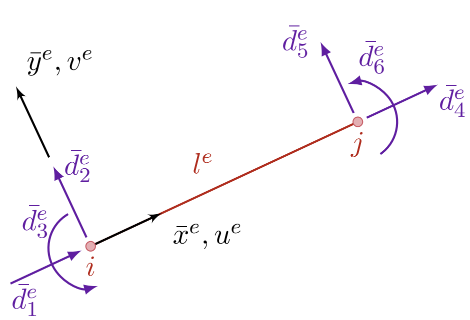
$$\bar{\mathbf{p}}^e = \int_0^{l^e} \mathbf{N}^{e\mathrm{T}} f^e \mathrm{d}\bar{x}^e$$

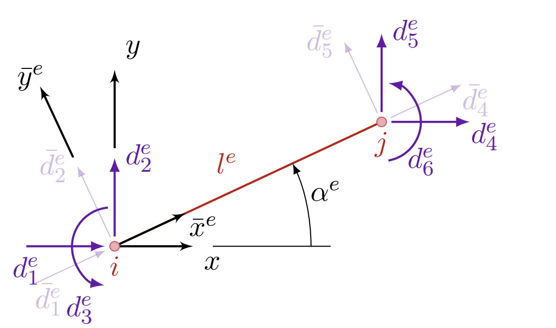
dla $f_y = \mathrm{const} = \widehat{f}_y$

$$\bar{\mathbf{p}}^e = \left\{ \frac{\widehat{f}_y l}{2},\ \frac{\widehat{f}_y l^2}{12},\ \frac{\widehat{f}_y l}{2},\ -\frac{\widehat{f}_y l^2}{12} \right\}^e$$



2.4.3. Element ramowy





Wektor funkcji przemieszczeń

$$\mathbf{u} = \{u(x), v(x)\}$$

Wektor uogólnionych odkształceń

$$\mathbf{e} = \{\varepsilon_x, \kappa\}$$

Wektor uogólnionych naprężeń

$$\mathbf{s} = \{N(x), M(x)\}$$

Wektor obciążenia po długości elementu

$$\mathbf{f} = \{f_x, f_y\}$$

Macierz związków konstytutywnych

$$\mathbf{D} = \begin{bmatrix} EA & 0 \\ 0 & EI \end{bmatrix}$$

Macierz operatorów różniczkowych

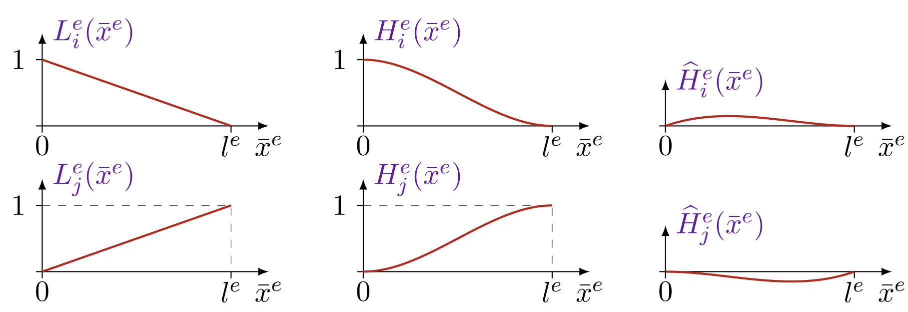
$$\mathbf{L} = \begin{bmatrix} \dfrac{\mathrm{d}}{\mathrm{d}x} & 0 \\ 0 & -\dfrac{\mathrm{d}^2}{\mathrm{d}x^2} \end{bmatrix}$$

Aproksymacja

$$\mathbf{u}^e(x) = \mathbf{N}^e(x)\bar{\mathbf{d}}^e$$

$$\mathbf{N}^e = \begin{bmatrix} L_i^e & 0 & 0 & L_j^e & 0 & 0 \\ 0 & H_i^e & \widehat{H}_i^e & 0 & H_j^e & \widehat{H}_j^e \end{bmatrix}$$

$$\bar{\mathbf{d}}^e = \begin{bmatrix} \bar{d}_1 \\ \bar{d}_2 \\ \bar{d}_3 \\ \bar{d}_4 \\ \bar{d}_5 \\ \bar{d}_6 \end{bmatrix}$$



Macierz sztywności

$$\bar{\mathbf{K}}^e = \int_0^{l^e} \mathbf{B}^{e\mathrm{T}} \mathbf{D}^e \mathbf{B}^e \mathrm{d}\bar{x}^e$$

$$\bar{\mathbf{K}}^e = \frac{EI}{l^3}\begin{bmatrix} \frac{Al^2}{I} & 0 & 0 & -\frac{Al^2}{I} & 0 & 0 \\ 0 & 12 & 6l & 0 & -12 & 6l \\ 0 & 6l & 4l^2 & 0 & -6l & 2l^2 \\ -\frac{Al^2}{I} & 0 & 0 & \frac{Al^2}{I} & 0 & 0 \\ 0 & -12 & -6l & 0 & 12 & -6l \\ 0 & 6l & 2l^2 & 0 & -6l & 4l^2 \end{bmatrix}^e$$

Macierz transformacji
$c = \cos(\alpha^e)$ **i** $s = \sin(\alpha^e)$

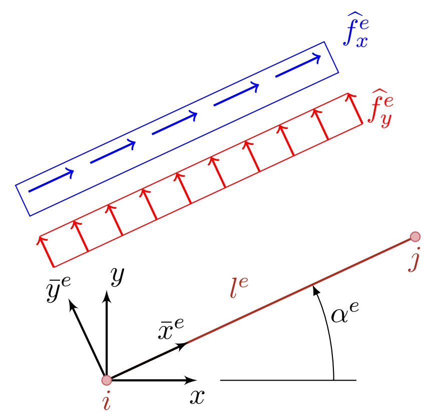
$$\mathbf{T}^e = \begin{bmatrix} c & s & 0 & 0 & 0 & 0 \\ -s & c & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & c & s & 0 \\ 0 & 0 & 0 & -s & c & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Wektor zastępników

$$\bar{\mathbf{p}}^e = \int_0^{l^e} \mathbf{N}^{e\mathrm{T}} \mathbf{f}^e \mathrm{d}\bar{x}^e$$

dla $f_x = \text{const} = \widehat{f}_x$ i $f_y = \text{const} = \widehat{f}_y$

$$\bar{\mathbf{p}}^e = \left\{ \frac{\widehat{f}_x l}{2},\ \frac{\widehat{f}_y l}{2},\ \frac{\widehat{f}_y l^2}{12},\ \frac{\widehat{f}_x l}{2},\ \frac{\widehat{f}_y l}{2},\ -\frac{\widehat{f}_y l^2}{12} \right\}^e$$



2.5. Zagadnienie 2D

Wektor funkcji przemieszczeń

$$\mathbf{u} = \{u(x,y), v(x,y)\}$$

Wektor odkształceń

$$\mathbf{e} = \{\varepsilon_x, \varepsilon_y, \gamma_{xy}\}$$

Wektor naprężeń

$$\mathbf{s} = \{\sigma_x, \sigma_y, \tau_{xy}\}$$

Wektor intensywności sił brzegowych

$$\mathbf{t} = \{t_x, t_y\}$$

Wektor intensywności sił powierzchniowych

$$\mathbf{f} = \{f_x, f_y\}$$

Macierz operatorów różniczkowych

$$\mathbf{L} = \begin{bmatrix} \dfrac{\partial}{\partial x} & 0 \\ 0 & \dfrac{\partial}{\partial y} \\ \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial x} \end{bmatrix}$$

Macierz związków konstytutywnych PSN: $\sigma_z = 0 \qquad \varepsilon_z = -\frac{\nu}{E}(\sigma_x + \sigma_y)$

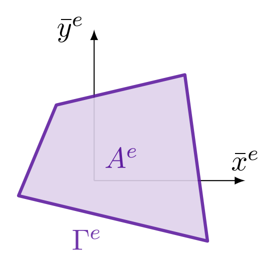
$$\mathbf{D} = \frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix}$$

Macierz związków konstytutywnych PSO: $\varepsilon_z = 0 \qquad \sigma_z = \nu(\sigma_x + \sigma_y)$

$$\mathbf{D} = \frac{E}{(1+\nu)(1-2\nu)}\begin{bmatrix} 1-\nu & \nu & 0 \\ \nu & 1-\nu & 0 \\ 0 & 0 & \frac{1-2\nu}{2} \end{bmatrix}$$

Macierz sztywności (dla PSO $h^e = 1$ m)

$$\bar{\mathbf{K}}^e = \int_{A^e} \mathbf{B}^{e\mathrm{T}} \mathbf{D}^e \mathbf{B}^e h^e \mathrm{d}A^e$$



Wektor obciążenia elementu (dla PSO $h^e = 1$ m)

$$\bar{\mathbf{p}}^e = \int_{A^e} \mathbf{N}^{e\mathrm{T}} \mathbf{f}^e h^e \mathrm{d}A^e$$

Wektor sił brzegowych (dla PSO $h^e = 1$ m)

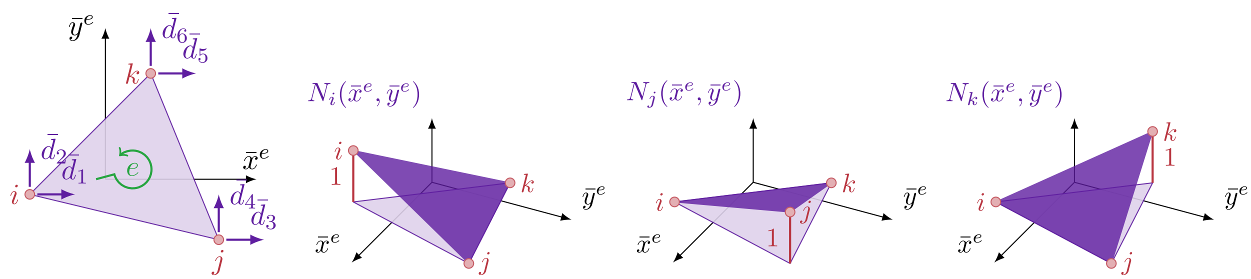
$$\bar{\mathbf{p}}^e_{\mathrm{b}} = \int_{\Gamma^e} \mathbf{N}^{e\mathrm{T}} \mathbf{t}^e h^e \mathrm{d}\Gamma^e$$

2.6. Elementy skończone 2D

Element trójwęzłowy

$$\mathbf{u}^e(x,y) = \mathbf{N}^e(x,y)\bar{\mathbf{d}}^e$$

$$\mathbf{N}^e = \begin{bmatrix} N_i^e & 0 & N_j^e & 0 & N_k^e & 0 \\ 0 & N_i^e & 0 & N_j^e & 0 & N_k^e \end{bmatrix}, \qquad \bar{\mathbf{d}}^e = \begin{bmatrix} \bar{d}_1 \\ \bar{d}_2 \\ \bar{d}_3 \\ \bar{d}_4 \\ \bar{d}_5 \\ \bar{d}_6 \end{bmatrix}$$



Wyznaczenie funkcji kształtu dla elementu trójwęzłowego

Funkcja kształtu $N_i(\bar{x}^e, \bar{y}^e) \Longrightarrow N_i(\bar{x}_i^e, \bar{y}_i^e) = 1,\ N_i(\bar{x}_j^e, \bar{y}_j^e) = 0,\ N_i(\bar{x}_k^e, \bar{y}_k^e) = 0$
$N_i(\bar{x}^e, \bar{y}^e) = \alpha_{1i} + \alpha_{2i}\bar{x}^e + \alpha_{3i}\bar{y}^e$

$$\begin{bmatrix} 1 & \bar{x}_i & \bar{y}_i \\ 1 & \bar{x}_j & \bar{y}_j \\ 1 & \bar{x}_k & \bar{y}_k \end{bmatrix} \begin{bmatrix} \alpha_{1i} \\ \alpha_{2i} \\ \alpha_{3i} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

rozwiązanie układu równań metodą wyznaczników

$$W = \left\| \begin{bmatrix} 1 & \bar{x}_i & \bar{y}_i \\ 1 & \bar{x}_j & \bar{y}_j \\ 1 & \bar{x}_k & \bar{y}_k \end{bmatrix} \right\| = 2P_\triangle$$

$$W_{\alpha_{1i}} = \left\| \begin{bmatrix} 1 & \bar{x}_i & \bar{y}_i \\ 0 & \bar{x}_j & \bar{y}_j \\ 0 & \bar{x}_k & \bar{y}_k \end{bmatrix} \right\| = \bar{x}_j\bar{y}_k - \bar{x}_k\bar{y}_j \qquad \Longrightarrow \qquad \alpha_{1i} = \frac{W_{\alpha_{1i}}}{W} = \frac{\bar{x}_j\bar{y}_k - \bar{x}_k\bar{y}_j}{2P_\triangle}$$

$$W_{\alpha_{2i}} = \left\| \begin{bmatrix} 1 & 1 & \bar{y}_i \\ 1 & 0 & \bar{y}_j \\ 1 & 0 & \bar{y}_k \end{bmatrix} \right\| = \bar{y}_j - \bar{y}_k, \qquad \Longrightarrow \qquad \alpha_{2i} = \frac{W_{\alpha_{2i}}}{W} = \frac{\bar{y}_j - \bar{y}_k}{2P_\triangle}$$

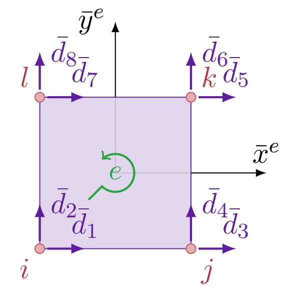
$$W_{\alpha_{3i}} = \left\| \begin{bmatrix} 1 & \bar{x}_i & 1 \\ 1 & \bar{x}_j & 0 \\ 1 & \bar{x}_k & 0 \end{bmatrix} \right\| = \bar{x}_k - \bar{x}_j \qquad \Longrightarrow \qquad \alpha_{3i} = \frac{W_{\alpha_{3i}}}{W} = \frac{\bar{x}_k - \bar{x}_j}{2P_\triangle}$$

$$N_i(\bar{x}, \bar{y}) = \frac{\bar{x}_j\bar{y}_k - \bar{x}_k\bar{y}_j + (\bar{y}_j - \bar{y}_k)\bar{x}^e + (\bar{x}_k - \bar{x}_j)\bar{y}^e}{2P_\triangle}$$

Element czterowęzłowy

$$\mathbf{u}^e(\bar{x},\bar{y}) = \mathbf{N}^e(\bar{x},\bar{y})\bar{\mathbf{d}}^e$$

$$\mathbf{N}^e = \begin{bmatrix} N_i^e & 0 & N_j^e & 0 & N_k^e & 0 & N_l^e & 0 \\ 0 & N_i^e & 0 & N_j^e & 0 & N_k^e & 0 & N_l^e \end{bmatrix}, \qquad \bar{\mathbf{d}}^e = \begin{bmatrix} \bar{d}_1 \\ \bar{d}_2 \\ \bar{d}_3 \\ \bar{d}_4 \\ \bar{d}_5 \\ \bar{d}_6 \\ \bar{d}_7 \\ \bar{d}_8 \end{bmatrix}$$



np. dla $N_i(\bar{x}^e,\bar{y}^e)$

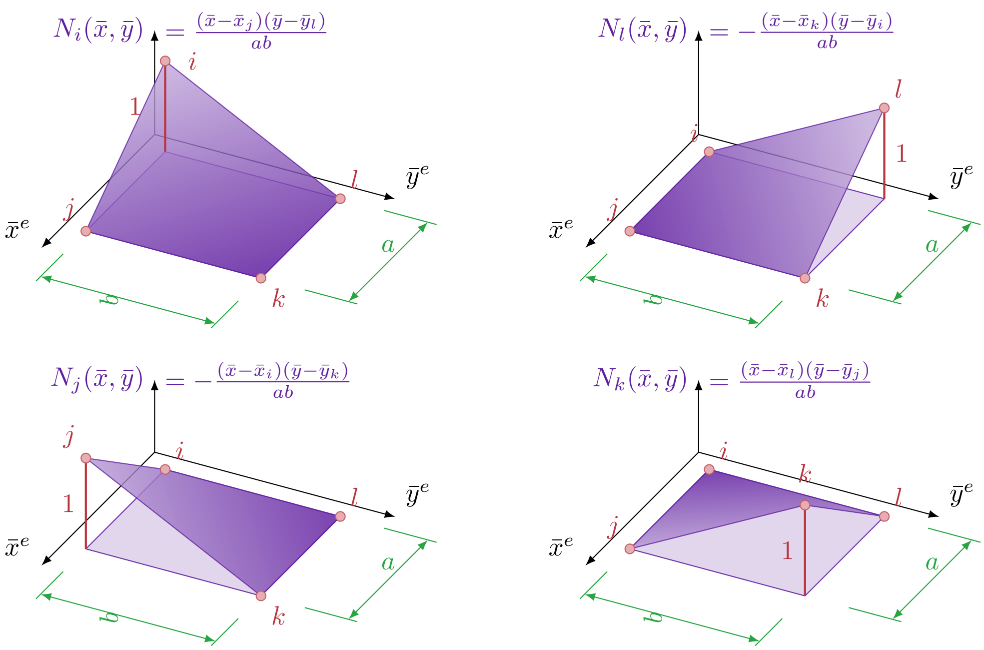
$N_i(\bar{x}_i^e,\bar{y}_i^e) = 1$

$N_i(\bar{x}_j^e,\bar{y}_j^e) = 0$

$N_i(\bar{x}_k^e,\bar{y}_k^e) = 0$

$N_i(\bar{x}_l^e,\bar{y}_l^e) = 0$

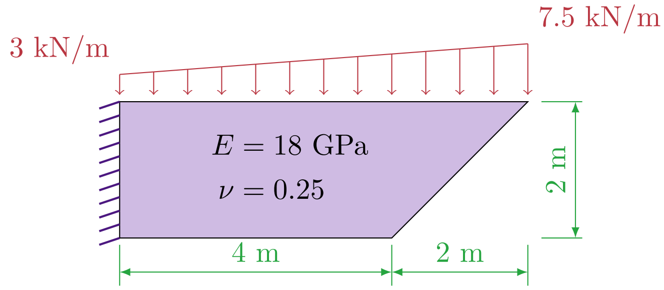
Element prostokątny

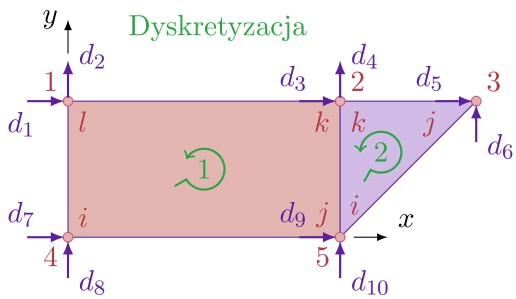


2.7. Przykłady

2.7.1. Zagadnienie PSO

1. Dane





nr. elem.	nr. węzł. elem.			
1	4	5	2	1
2	5	3	2	

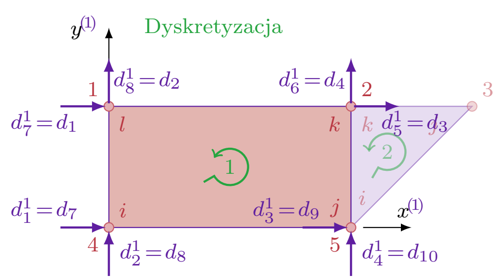
2. Macierz związków konstytutywnych

$$\mathbf{D} = \frac{18 \cdot 10^6}{(1+0.25)(1-2\cdot 0.25)} \begin{bmatrix} 1-0.25 & 0.25 & 0 \\ 0.25 & 1-0.25 & 0 \\ 0 & 0 & \frac{1-2\cdot 0.25}{2} \end{bmatrix} \text{ [kPa]}$$

$$\mathbf{D} = \begin{bmatrix} 21.6 & 7.2 & 0 \\ 7.2 & 21.6 & 0 \\ 0 & 0 & 7.2 \end{bmatrix} \cdot 10^6 \text{ [kPa]}$$

3. Wyznaczenie macierzy funkcji kształtu $\mathbf{N}^e$ i macierzy sztywności $\mathbf{K}^e$

   - Element 1



$$N_i^1(x^{(1)}, y^{(1)}) = \frac{x^{(1)}y^{(1)} - 2x^{(1)} - 4y^{(1)} + 8}{8}, \qquad N_k^1(x^{(1)}, y^{(1)}) = \frac{x^{(1)}y^{(1)}}{8}$$

$$N_j^1(x^{(1)}, y^{(1)}) = \frac{2x^{(1)} - x^{(1)}y^{(1)}}{8}, \qquad N_l^1(x^{(1)}, y^{(1)}) = \frac{4y^{(1)} - x^{(1)}y^{(1)}}{8}$$

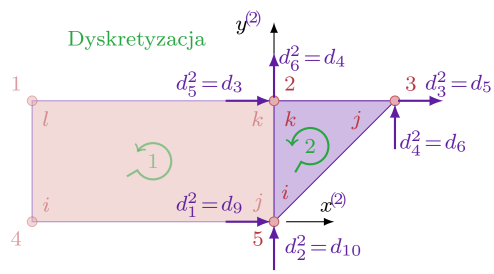
$$\mathbf{N}^1 = \begin{bmatrix} N_i^1 & 0 & N_j^1 & 0 & N_k^1 & 0 & N_l^1 & 0 \\ 0 & N_i^1 & 0 & N_j^1 & 0 & N_k^1 & 0 & N_l^1 \end{bmatrix}$$

$$\mathbf{B}^e(x^e, y^e) = \begin{bmatrix} \frac{\mathrm{d}N_i^e}{\mathrm{d}x} & 0 & \frac{\mathrm{d}N_j^e}{\mathrm{d}x} & 0 & \frac{\mathrm{d}N_k^e}{\mathrm{d}x} & 0 & \frac{\mathrm{d}N_l^e}{\mathrm{d}x} & 0 \\ 0 & \frac{\mathrm{d}N_i^e}{\mathrm{d}y} & 0 & \frac{\mathrm{d}N_j^e}{\mathrm{d}y} & 0 & \frac{\mathrm{d}N_k^e}{\mathrm{d}y} & 0 & \frac{\mathrm{d}N_l^e}{\mathrm{d}y} \\ \frac{\mathrm{d}N_i^e}{\mathrm{d}y} & \frac{\mathrm{d}N_i^e}{\mathrm{d}x} & \frac{\mathrm{d}N_j^e}{\mathrm{d}y} & \frac{\mathrm{d}N_j^e}{\mathrm{d}x} & \frac{\mathrm{d}N_k^e}{\mathrm{d}y} & \frac{\mathrm{d}N_k^e}{\mathrm{d}x} & \frac{\mathrm{d}N_l^e}{\mathrm{d}y} & \frac{\mathrm{d}N_l^e}{\mathrm{d}x} \end{bmatrix}$$

$$\mathbf{B}^1(x^{(1)}, y^{(1)}) = \begin{bmatrix} \frac{y^{(1)}}{8} - \frac{1}{4} & 0 & \frac{1}{4} - \frac{y^{(1)}}{8} & 0 & \frac{y^{(1)}}{8} & 0 & -\frac{y^{(1)}}{8} & 0 \\ 0 & \frac{x^{(1)}}{8} - \frac{1}{2} & 0 & -\frac{x^{(1)}}{8} & 0 & \frac{x^{(1)}}{8} & 0 & \frac{1}{2} - \frac{x^{(1)}}{8} \\ \frac{x^{(1)}}{8} - \frac{1}{2} & \frac{y^{(1)}}{8} - \frac{1}{4} & -\frac{x^{(1)}}{8} & \frac{1}{4} - \frac{y^{(1)}}{8} & \frac{x^{(1)}}{8} & \frac{y^{(1)}}{8} & \frac{1}{2} - \frac{x^{(1)}}{8} & -\frac{y^{(1)}}{8} \end{bmatrix}$$

$$\mathbf{K}^1 = \int_0^2 \int_0^4 \mathbf{B}^{1\,\mathrm{T}} \mathbf{D} \mathbf{B}^1 \ \mathrm{d}x^{(1)} \mathrm{d}y^{(1)} = \begin{bmatrix} 84 & 36 & -12 & 0 & -42 & -36 & -30 & 0 \\ 36 & 156 & 0 & 60 & -36 & -78 & 0 & -138 \\ -12 & 0 & 84 & -36 & -30 & 0 & -42 & 36 \\ 0 & 60 & -36 & 156 & 0 & -138 & 36 & -78 \\ -42 & -36 & -30 & 0 & 84 & 36 & -12 & 0 \\ -36 & -78 & 0 & -138 & 36 & 156 & 0 & 60 \\ -30 & 0 & -42 & 36 & -12 & 0 & 84 & -36 \\ 0 & -138 & 36 & -78 & 0 & 60 & -36 & 156 \end{bmatrix} \cdot 10^5$$

- Element 2



$$N_i^2(x^{(2)}, y^{(2)}) = \frac{2 - y^{(2)}}{2}, \qquad N_k^2(x^{(2)}, y^{(2)}) = \frac{y^{(2)} - x^{(2)}}{2}$$

$$N_j^2(x^{(2)}, y^{(2)}) = \frac{x^{(2)}}{2}$$

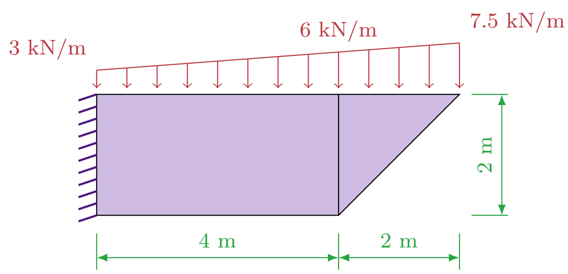
$$\mathbf{N}^2 = \begin{bmatrix} N_i^2 & 0 & N_j^2 & 0 & N_k^2 & 0 \\ 0 & N_i^2 & 0 & N_j^2 & 0 & N_k^2 \end{bmatrix}$$

$$\mathbf{B}^e(x^e, y^e) = \begin{bmatrix} \frac{\mathrm{d}N_i^e}{\mathrm{d}x} & 0 & \frac{\mathrm{d}N_j^e}{\mathrm{d}x} & 0 & \frac{\mathrm{d}N_k^e}{\mathrm{d}x} & 0 \\ 0 & \frac{\mathrm{d}N_i^e}{\mathrm{d}y} & 0 & \frac{\mathrm{d}N_j^e}{\mathrm{d}y} & 0 & \frac{\mathrm{d}N_k^e}{\mathrm{d}y} \\ \frac{\mathrm{d}N_i^e}{\mathrm{d}y} & \frac{\mathrm{d}N_i^e}{\mathrm{d}x} & \frac{\mathrm{d}N_j^e}{\mathrm{d}y} & \frac{\mathrm{d}N_j^e}{\mathrm{d}x} & \frac{\mathrm{d}N_k^e}{\mathrm{d}y} & \frac{\mathrm{d}N_k^e}{\mathrm{d}x} \end{bmatrix}$$

$$\mathbf{B}^2(x^{(2)}, y^{(2)}) = \begin{bmatrix} 0 & 0 & \frac{1}{2} & 0 & -\frac{1}{2} & 0 \\ 0 & -\frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} \\ -\frac{1}{2} & 0 & 0 & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \end{bmatrix}$$

$$\mathbf{K}^2 = \mathbf{B}^{2^\mathrm{T}} \mathbf{D} \mathbf{B}^2 A^2 = \begin{bmatrix} 36 & 0 & 0 & -36 & -36 & 36 \\ 0 & 108 & -36 & 0 & 36 & -108 \\ 0 & -36 & 108 & 0 & -108 & 36 \\ -36 & 0 & 0 & 36 & 36 & -36 \\ -36 & 36 & -108 & 36 & 144 & -72 \\ 36 & -108 & 36 & -36 & -72 & 144 \end{bmatrix} \cdot 10^5$$

4. Wektor $\mathbf{p}_\mathrm{b}$



- Element 1

$$\mathbf{p}_\mathrm{b}^1 = \int_{\Gamma_{ij}^1} \mathbf{N}^{1^\mathrm{T}} \overset{w.b. = 0}{\mathbf{t}} \, \mathrm{d}\Gamma + \int_{\Gamma_{jk}^1} \mathbf{N}^{1^\mathrm{T}} \mathbf{t} \mathrm{d}\Gamma + \int_{\Gamma_{kl}^1} \mathbf{N}^{1^\mathrm{T}} \mathbf{t} \mathrm{d}\Gamma + \int_{\Gamma_{li}^1} \mathbf{N}^{1^\mathrm{T}} \mathbf{t} \mathrm{d}\Gamma$$

wspólna krawędź – równowaga sił wzdłuż linii 2-5: $\mathbf{t}_{jk}^1 = -\mathbf{t}_{ki}^2$

$$= \int_0^4 \left(\mathbf{N}^1(x^{(1)}, y^{(1)} = 2) \right)^\mathrm{T} \begin{bmatrix} 0 \\ -3\left(1 - \frac{x^{(1)}}{4}\right) - 6\frac{x^{(1)}}{4} \end{bmatrix} \mathrm{d}x^{(1)}$$

$$+ \int_0^2 \left(\mathbf{N}^1(x^{(1)} = 0, y^{(1)}) \right)^\mathrm{T} \mathbf{t} \mathrm{d}y^{(1)}$$

$$= \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -10 \\ 0 \\ -8 \end{bmatrix} + \begin{bmatrix} r_1^1 \\ r_2^1 \\ 0 \\ 0 \\ 0 \\ 0 \\ r_7^1 \\ r_8^1 \end{bmatrix}$$

- Element 2

wspólna krawędź – równowaga sił wzdłuż linii 2-5: $\mathbf{t}^1_{jk} = -\mathbf{t}^2_{ki}$

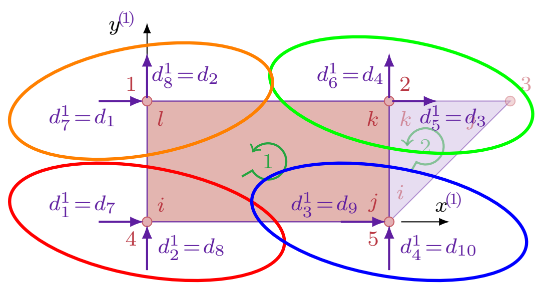
$$\mathbf{p}^2_b = \int_{\Gamma^2_{ij}} \mathbf{N}^{2^\mathrm{T}} \underbrace{\mathbf{t}}_{w.b.\,=\,0} \mathrm{d}\Gamma + \int_{\Gamma^2_{jk}} \mathbf{N}^{2^\mathrm{T}} \mathbf{t}\mathrm{d}\Gamma + \int_{\Gamma^2_{ki}} \mathbf{N}^{2^\mathrm{T}} \mathbf{t}\mathrm{d}\Gamma$$

$$= \int_0^2 \left(\mathbf{N}^2(x^{(2)}, y^{(2)} = 2)\right)^\mathrm{T} \begin{bmatrix} 0 \\ -6\left(1-\frac{x^{(2)}}{2}\right) - 7.5\frac{x^{(2)}}{2} \end{bmatrix} \mathrm{d}x^{(2)}$$

$$= \begin{bmatrix} 0 \\ 0 \\ 0 \\ -7 \\ 0 \\ -6.5 \end{bmatrix}$$

5. Agregacja – Macierze Boole'a $\mathbb{B}$

- Element 1

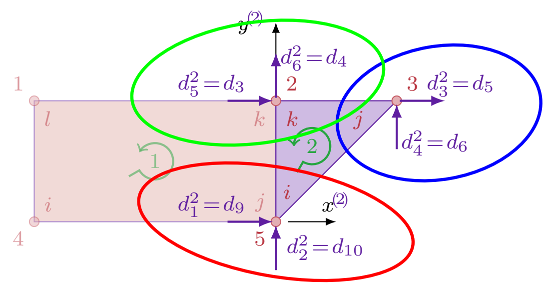


nr. lok.

$$\mathbb{B}^1 = \begin{bmatrix} 0&0&0&0&0&0&1&0&0&0 \\ 0&0&0&0&0&0&0&1&0&0 \\ 0&0&0&0&0&0&0&0&1&0 \\ 0&0&0&0&0&0&0&0&0&1 \\ 0&0&1&0&0&0&0&0&0&0 \\ 0&0&0&1&0&0&0&0&0&0 \\ 1&0&0&0&0&0&0&0&0&0 \\ 0&1&0&0&0&0&0&0&0&0 \end{bmatrix} \begin{matrix} ① \\ ② \\ ③ \\ ④ \\ ⑤ \\ ⑥ \\ ⑦ \\ ⑧ \end{matrix}$$

nr. glob. ① ② ③ ④ ⑤ ⑥ ⑦ ⑧ ⑨ ⑩

- Element 2



nr. lok.

$$\mathbb{B}^2 = \begin{bmatrix} 0&0&0&0&0&0&0&0&1&0 \\ 0&0&0&0&0&0&0&0&0&1 \\ 0&0&0&0&1&0&0&0&0&0 \\ 0&0&0&0&0&1&0&0&0&0 \\ 0&0&1&0&0&0&0&0&0&0 \\ 0&0&0&1&0&0&0&0&0&0 \end{bmatrix} \begin{matrix} ① \\ ② \\ ③ \\ ④ \\ ⑤ \\ ⑥ \end{matrix}$$

nr. glob. ① ② ③ ④ ⑤ ⑥ ⑦ ⑧ ⑨ ⑩

6. Agregacja - Macierz sztywności

$$\mathbf{K} = \mathbb{B}^{1^\mathrm{T}} \mathbf{K}^1 \mathbb{B}^1 + \mathbb{B}^{2^\mathrm{T}} \mathbf{K}^2 \mathbb{B}^2$$

$$\mathbf{K} = \begin{bmatrix} 84 & -36 & -12 & 0 & 0 & 0 & -30 & 0 & -42 & 36 \\ -36 & 156 & 0 & 60 & 0 & 0 & 0 & -138 & 36 & -78 \\ -12 & 0 & 228 & -36 & -108 & 36 & -42 & -36 & -66 & 36 \\ 0 & 60 & -36 & 300 & 36 & -36 & -36 & -78 & 36 & -246 \\ 0 & 0 & -108 & 36 & 108 & 0 & 0 & 0 & 0 & -36 \\ 0 & 0 & 36 & -36 & 0 & 36 & 0 & 0 & -36 & 0 \\ -30 & 0 & -42 & -36 & 0 & 0 & 84 & 36 & -12 & 0 \\ 0 & -138 & -36 & -78 & 0 & 0 & 36 & 156 & 0 & 60 \\ -42 & 36 & -66 & 36 & 0 & -36 & -12 & 0 & 120 & -36 \\ 36 & -78 & 36 & -246 & -36 & 0 & 0 & 60 & -36 & 264 \end{bmatrix} \cdot 10^5$$

7. Agregacja - Wektor obciążenia

$$\mathbf{p}_{\mathrm{b}} = \mathbb{B}^{1^{\mathrm{T}}} \mathbf{p}_{\mathrm{b}}^{1} + \mathbb{B}^{2^{\mathrm{T}}} \mathbf{p}_{\mathrm{b}}^{2}, \qquad \mathbf{p} = \mathbf{0}$$

$$\mathbf{p}_{\mathrm{b}} = \begin{bmatrix} 0 \\ -8 \\ 0 \\ -16.5 \\ 0 \\ -7 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} r_7^1 = r_1 \\ r_8^1 = r_2 \\ 0 \\ 0 \\ 0 \\ 0 \\ r_1^1 = r_7 \\ r_2^1 = r_8 \\ 0 \\ 0 \end{bmatrix}$$

8. Układ równań MES: $\mathbf{Kd} = \mathbf{p} + \mathbf{p}_b$

$$\begin{bmatrix} 84 & -36 & -12 & 0 & 0 & 0 & -30 & 0 & -42 & 36 \\ -36 & 156 & 0 & 60 & 0 & 0 & 0 & -138 & 36 & -78 \\ -12 & 0 & 228 & -36 & -108 & 36 & -42 & -36 & -66 & 36 \\ 0 & 60 & -36 & 300 & 36 & -36 & -36 & -78 & 36 & -246 \\ 0 & 0 & -108 & 36 & 108 & 0 & 0 & 0 & 0 & -36 \\ 0 & 0 & 36 & -36 & 0 & 36 & 0 & 0 & -36 & 0 \\ -30 & 0 & -42 & -36 & 0 & 0 & 84 & 36 & -12 & 0 \\ 0 & -138 & -36 & -78 & 0 & 0 & 36 & 156 & 0 & 60 \\ -42 & 36 & -66 & 36 & 0 & -36 & -12 & 0 & 120 & -36 \\ 36 & -78 & 36 & -246 & -36 & 0 & 0 & 60 & -36 & 264 \end{bmatrix} \cdot 10^5 \begin{bmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \\ d_5 \\ d_6 \\ d_7 \\ d_8 \\ d_9 \\ d_{10} \end{bmatrix} = \begin{bmatrix} 0 \\ -8 \\ 0 \\ -16.5 \\ 0 \\ -7 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} r_1 \\ r_2 \\ 0 \\ 0 \\ 0 \\ 0 \\ r_7 \\ r_8 \\ 0 \\ 0 \end{bmatrix}$$

9. Uwzględnienie warunków brzegowych

$$\begin{bmatrix} 84 & -36 & -12 & 0 & 0 & 0 & -30 & 0 & -42 & 36 \\ -36 & 156 & 0 & 60 & 0 & 0 & 0 & -138 & 36 & -78 \\ -12 & 0 & 228 & -36 & -108 & 36 & -42 & -36 & -66 & 36 \\ 0 & 60 & -36 & 300 & 36 & -36 & -36 & -78 & 36 & -246 \\ 0 & 0 & -108 & 36 & 108 & 0 & 0 & 0 & 0 & -36 \\ 0 & 0 & 36 & -36 & 0 & 36 & 0 & 0 & -36 & 0 \\ -30 & 0 & -42 & -36 & 0 & 0 & 84 & 36 & -12 & 0 \\ 0 & -138 & -36 & -78 & 0 & 0 & 36 & 156 & 0 & 60 \\ -42 & 36 & -66 & 36 & 0 & -36 & -12 & 0 & 120 & -36 \\ 36 & -78 & 36 & -246 & -36 & 0 & 0 & 60 & -36 & 264 \end{bmatrix} \cdot 10^5 \begin{bmatrix} 0 \\ 0 \\ d_3 \\ d_4 \\ d_5 \\ d_6 \\ 0 \\ 0 \\ d_9 \\ d_{10} \end{bmatrix} = \begin{bmatrix} 0 \\ -8 \\ 0 \\ -16.5 \\ 0 \\ -7 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} r_1 \\ r_2 \\ 0 \\ 0 \\ 0 \\ 0 \\ r_7 \\ r_8 \\ 0 \\ 0 \end{bmatrix}$$

Rozwiązanie:

$$\mathbf{d} = \{0\ 0\ 0.743\ \text{-}2.131\ 0.760\ \text{-}3.777\ 0\ 0\ \text{-}0.709\ \text{-}2.080\} \cdot 10^{-5}\ \text{m}$$

$$\mathbf{r} = \{\text{-}54\ 16.851\ 0\ 0\ 0\ 0\ 54\ 14.649\ 0\ 0\}\ \text{kN}$$

10. Powrót do elementu

- Element 1

$$\mathbf{d}^1 = \mathbb{B}^1\mathbf{d} = \{0\ 0\ \text{-}0.709\ \text{-}2.080\ 0.743\ \text{-}2.131\ 0\ 0\} \cdot 10^{-5}$$

$$\mathbf{e}^1 = \mathbf{B}^1\mathbf{d}^1$$

$$\mathbf{e}^1 = \begin{bmatrix} 1.815y - 1.773 \\ -0.064x \\ 1.815x - 0.064y - 5.199 \end{bmatrix} \cdot 10^{-6}, \qquad \mathbf{e}^1(2,1) = \begin{bmatrix} 0.424 \\ -1.272 \\ -16.319 \end{bmatrix} \cdot 10^{-7}$$

$$\mathbf{s}^1 = \mathbf{D}\mathbf{e}^1$$

$$\mathbf{s}^1 = \begin{bmatrix} 39.214y - 0.458x - 38.299 \\ 13.071y - 1.373x - 12.766 \\ 13.071x - 0.458y - 37.435 \end{bmatrix}, \qquad \mathbf{s}^1(2,1) = \begin{bmatrix} 0 \\ -2.442 \\ -11.750 \end{bmatrix}\ \text{kPa}$$

$$\sigma_z^1 = \nu(\sigma_x^1 + \sigma_y^1) = 0.25(0 - 2.442) = -0.615\ \text{kPa}$$

- Element 2

$$\mathbf{d}^2 = \mathbb{B}^2\mathbf{d} = \{\text{-}0.709\ \text{-}2.080\ 0.760\ \text{-}3.777\ 0.743\ \text{-}2.131\} \cdot 10^{-5}$$

$$\mathbf{e}^2 = \mathbf{B}^2\mathbf{d}^2$$

$$\mathbf{e}^2 = \begin{bmatrix} 0.848 \\ -2.543 \\ -9.722 \end{bmatrix} \cdot 10^{-7}$$

$$\mathbf{s}^2 = \begin{bmatrix} 0 \\ -4.883 \\ -7.000 \end{bmatrix}\ \text{kPa}, \qquad \sigma_z^2 = \nu(\sigma_x^2 + \sigma_y^2) = 0.25(0 - 4.883) = -1.221\ \text{kPa}$$

11. Wyznaczenie wartości przemieszczeń w wybranych punktach

- Element 1 – punkt $(x^{(1)} = 2, y^{(1)} = 1)$

$$\mathbf{u}^1(x^{(1)}, y^{(1)}) = \mathbf{N}^1(x^{(1)}, y^{(1)})\mathbf{d}^1$$

$$\mathbf{u}^1(2,1) = \begin{bmatrix} \frac{1}{4} & 0 & \frac{1}{4} & 0 & \frac{1}{4} & 0 & \frac{1}{4} & 0 \\ 0 & \frac{1}{4} & 0 & \frac{1}{4} & 0 & \frac{1}{4} & 0 & \frac{1}{4} \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 0 \\ -0.709 \\ -2.080 \\ 0.743 \\ -2.131 \\ 0 \\ 0 \end{bmatrix} \cdot 10^{-5} = \begin{bmatrix} 0.0085 \\ -1.0526 \end{bmatrix} \cdot 10^{-5} \text{ [m]}$$

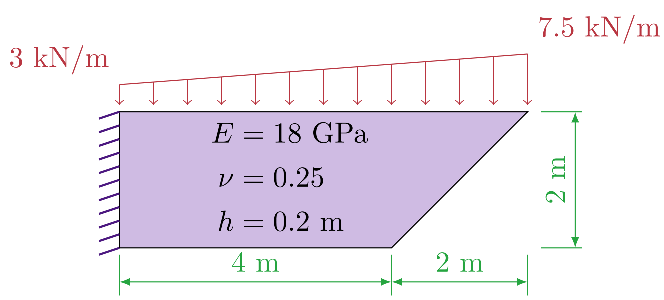
- Element 2 – punkt $(x^{(2)} = 0.5, y^{(2)} = 1.5)$

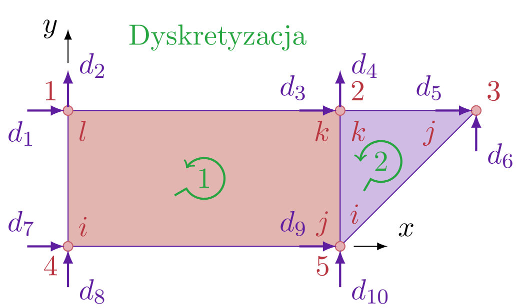
$$\mathbf{u}^2(x^{(2)}, y^{(2)}) = \mathbf{N}^2(x^{(2)}, y^{(2)})\mathbf{d}^2$$

$$\mathbf{u}^2(0.5, 1.5) = \begin{bmatrix} \frac{1}{4} & 0 & \frac{1}{4} & 0 & \frac{1}{2} & 0 \\ 0 & \frac{1}{4} & 0 & \frac{1}{4} & 0 & \frac{1}{2} \end{bmatrix} \cdot \begin{bmatrix} -0.709 \\ -2.080 \\ 0.760 \\ -3.777 \\ 0.743 \\ -2.131 \end{bmatrix} \cdot 10^{-5} = \begin{bmatrix} 0.3843 \\ -2.5296 \end{bmatrix} \cdot 10^{-5} \text{ [m]}$$

2.7.2. Statyka tarczy

1. Dane





nr. elem.	nr. węzł. elem.			
1	4	5	2	1
2	5	3	2	

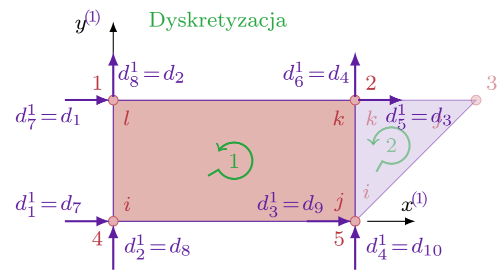
2. Macierz związków konstytutywnych

$$\mathbf{D} = \frac{18 \cdot 10^6}{1 - 0.25^2} \begin{bmatrix} 1 & 0.25 & 0 \\ 0.25 & 1 & 0 \\ 0 & 0 & \frac{1-0.25}{2} \end{bmatrix} \text{ [kPa]}$$

$$\mathbf{D} = \begin{bmatrix} 19.2 & 4.8 & 0 \\ 4.8 & 19.2 & 0 \\ 0 & 0 & 7.2 \end{bmatrix} \cdot 10^6 \text{ [kPa]}$$

3. Wyznaczenie macierzy funkcji kształtu $\mathbf{N}^e$ i macierzy sztywności $\mathbf{K}^e$

 - Element 1



$$N_i^1(x^{(1)}, y^{(1)}) = \frac{x^{(1)}y^{(1)} - 2x^{(1)} - 4y^{(1)} + 8}{8}, \qquad N_k^1(x^{(1)}, y^{(1)}) = \frac{x^{(1)}y^{(1)}}{8}$$

$$N_j^1(x^{(1)}, y^{(1)}) = \frac{2x^{(1)} - x^{(1)}y^{(1)}}{8}, \qquad N_l^1(x^{(1)}, y^{(1)}) = \frac{4y^{(1)} - x^{(1)}y^{(1)}}{8}$$

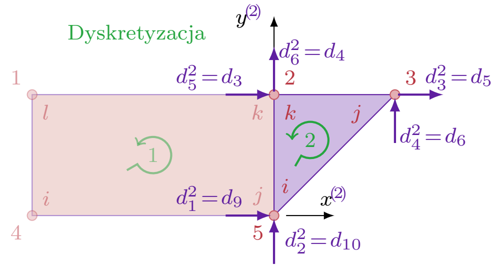
$$\mathbf{N}^1 = \begin{bmatrix} N_i^1 & 0 & N_j^1 & 0 & N_k^1 & 0 & N_l^1 & 0 \\ 0 & N_i^1 & 0 & N_j^1 & 0 & N_k^1 & 0 & N_l^1 \end{bmatrix}$$

$$\mathbf{B}^e(x^e, y^e) = \begin{bmatrix} \frac{\mathrm{d}N_i^e}{\mathrm{d}x} & 0 & \frac{\mathrm{d}N_j^e}{\mathrm{d}x} & 0 & \frac{\mathrm{d}N_k^e}{\mathrm{d}x} & 0 & \frac{\mathrm{d}N_l^e}{\mathrm{d}x} & 0 \\ 0 & \frac{\mathrm{d}N_i^e}{\mathrm{d}y} & 0 & \frac{\mathrm{d}N_j^e}{\mathrm{d}y} & 0 & \frac{\mathrm{d}N_k^e}{\mathrm{d}y} & 0 & \frac{\mathrm{d}N_l^e}{\mathrm{d}y} \\ \frac{\mathrm{d}N_i^e}{\mathrm{d}y} & \frac{\mathrm{d}N_i^e}{\mathrm{d}x} & \frac{\mathrm{d}N_j^e}{\mathrm{d}y} & \frac{\mathrm{d}N_j^e}{\mathrm{d}x} & \frac{\mathrm{d}N_k^e}{\mathrm{d}y} & \frac{\mathrm{d}N_k^e}{\mathrm{d}x} & \frac{\mathrm{d}N_l^e}{\mathrm{d}y} & \frac{\mathrm{d}N_l^e}{\mathrm{d}x} \end{bmatrix}$$

$$\mathbf{B}^1(x^{(1)}, y^{(1)}) = \begin{bmatrix} \frac{y^{(1)}}{8} - \frac{1}{4} & 0 & \frac{1}{4} - \frac{y^{(1)}}{8} & 0 & \frac{y^{(1)}}{8} & 0 & -\frac{y^{(1)}}{8} & 0 \\ 0 & \frac{x^{(1)}}{8} - \frac{1}{2} & 0 & -\frac{x^{(1)}}{8} & 0 & \frac{x^{(1)}}{8} & 0 & \frac{1}{2} - \frac{x^{(1)}}{8} \\ \frac{x^{(1)}}{8} - \frac{1}{2} & \frac{y^{(1)}}{8} - \frac{1}{4} & -\frac{x^{(1)}}{8} & \frac{1}{4} - \frac{y^{(1)}}{8} & \frac{x^{(1)}}{8} & \frac{y^{(1)}}{8} & \frac{1}{2} - \frac{x^{(1)}}{8} & -\frac{y^{(1)}}{8} \end{bmatrix}$$

$$\mathbf{K}^1 = \int_0^2 \int_0^4 \mathbf{B}^{1^\mathrm{T}} \mathbf{D} \mathbf{B}^1 h \, \mathrm{d}x^{(1)} \mathrm{d}y^{(1)} = \begin{bmatrix} 16 & 6 & -1.6 & -1.2 & -8 & -6 & -6.4 & 1.2 \\ 6 & 28 & 1.2 & 10.4 & -6 & -14 & -1.2 & -24.4 \\ -1.6 & 1.2 & 16 & -6 & -6.4 & -1.2 & -8 & 6 \\ -1.2 & 10.4 & -6 & 28 & 1.2 & -24.4 & 6 & -14 \\ -8 & -6 & -6.4 & 1.2 & 16 & 6 & -1.6 & -1.2 \\ -6 & -14 & -1.2 & -24.4 & 6 & 28 & 1.2 & 10.4 \\ -6.4 & -1.2 & -8 & 6 & -1.6 & 1.2 & 16 & -6 \\ 1.2 & -24.4 & 6 & -14 & -1.2 & 10.4 & -6 & 28 \end{bmatrix} \cdot 10^5$$

 - Element 2



$$N_i^2(x^{(2)}, y^{(2)}) = \frac{2 - y^{(2)}}{2}, \qquad N_k^2(x^{(2)}, y^{(2)}) = \frac{y^{(2)} - x^{(2)}}{2}$$

$$N_j^2(x^{(2)}, y^{(2)}) = \frac{x^{(2)}}{2}$$

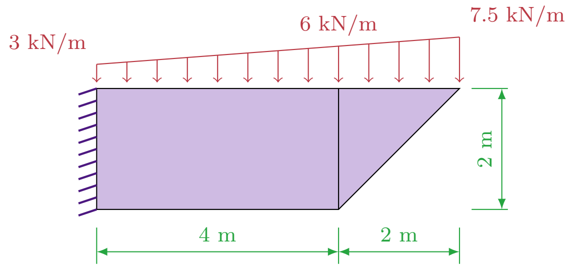
$$\mathbf{N}^2 = \begin{bmatrix} N_i^2 & 0 & N_j^2 & 0 & N_k^2 & 0 \\ 0 & N_i^2 & 0 & N_j^2 & 0 & N_k^2 \end{bmatrix}$$

$$\mathbf{B}^e(x^e, y^e) = \begin{bmatrix} \frac{\mathrm{d}N_i^e}{\mathrm{d}x} & 0 & \frac{\mathrm{d}N_j^e}{\mathrm{d}x} & 0 & \frac{\mathrm{d}N_k^e}{\mathrm{d}x} & 0 \\ 0 & \frac{\mathrm{d}N_i^e}{\mathrm{d}y} & 0 & \frac{\mathrm{d}N_j^e}{\mathrm{d}y} & 0 & \frac{\mathrm{d}N_k^e}{\mathrm{d}y} \\ \frac{\mathrm{d}N_i^e}{\mathrm{d}y} & \frac{\mathrm{d}N_i^e}{\mathrm{d}x} & \frac{\mathrm{d}N_j^e}{\mathrm{d}y} & \frac{\mathrm{d}N_j^e}{\mathrm{d}x} & \frac{\mathrm{d}N_k^e}{\mathrm{d}y} & \frac{\mathrm{d}N_k^e}{\mathrm{d}x} \end{bmatrix}$$

$$\mathbf{B}^2(x^{(2)}, y^{(2)}) = \begin{bmatrix} 0 & 0 & \frac{1}{2} & 0 & -\frac{1}{2} & 0 \\ 0 & -\frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} \\ -\frac{1}{2} & 0 & 0 & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \end{bmatrix}$$

$$\mathbf{K}^2 = \mathbf{B}^{2^{\mathrm{T}}} \mathbf{D} \mathbf{B}^2 h A^2 = \begin{bmatrix} 7.2 & 0 & 0 & -7.2 & -7.2 & 7.2 \\ 0 & 19.2 & -4.8 & 0 & 4.8 & -19.2 \\ 0 & -4.8 & 19.2 & 0 & -19.2 & 4.8 \\ -7.2 & 0 & 0 & 7.2 & 7.2 & -7.2 \\ -7.2 & 4.8 & -19.2 & 7.2 & 26.4 & -12 \\ 7.2 & -19.2 & 4.8 & -7.2 & -12 & 26.4 \end{bmatrix} \cdot 10^5$$

4. Wektor $\mathbf{p}_\mathrm{b}$



- Element 1

w.b. = 0

$$\mathbf{p}_\mathrm{b}^1 = \int_{\Gamma_{ij}^1} \mathbf{N}^{1\,\mathrm{T}} \mathbf{t}\,\mathrm{d}\Gamma + \int_{\Gamma_{jk}^1} \mathbf{N}^{1\,\mathrm{T}} \mathbf{t}\,\mathrm{d}\Gamma + \int_{\Gamma_{kl}^1} \mathbf{N}^{1\,\mathrm{T}} \mathbf{t}\,\mathrm{d}\Gamma + \int_{\Gamma_{li}^1} \mathbf{N}^{1\,\mathrm{T}} \mathbf{t}\,\mathrm{d}\Gamma$$

wspólna krawędź – równowaga sił wzdłuż linii 2-5: $\mathbf{t}_{jk}^1 = -\mathbf{t}_{ki}^2$

$$= \int_0^4 \left(\mathbf{N}^1(x^{(1)}, y^{(1)} = 2)\right)^\mathrm{T} \begin{bmatrix} 0 \\ -3\left(1 - \frac{x^{(1)}}{4}\right) - 6\frac{x^{(1)}}{4} \end{bmatrix} \mathrm{d}x^{(1)}$$

$$+ \int_0^2 \left(\mathbf{N}^1(x^{(1)} = 0, y^{(1)})\right)^\mathrm{T} \mathbf{t}\,\mathrm{d}y^{(1)}$$

$$= \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -10 \\ 0 \\ -8 \end{bmatrix} + \begin{bmatrix} r_1^1 \\ r_2^1 \\ 0 \\ 0 \\ 0 \\ 0 \\ r_7^1 \\ r_8^1 \end{bmatrix}$$

- Element 2

wspólna krawędź – równowaga sił wzdłuż linii 2-5: $\mathbf{t}_{jk}^1 = -\mathbf{t}_{ki}^2$

w.b. = 0

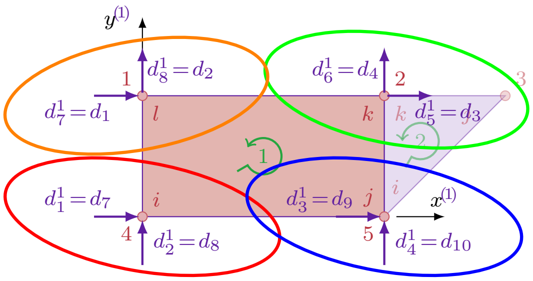
$$\mathbf{p}_b^2 = \int_{\Gamma_{ij}^2} \mathbf{N}^{2\,\mathrm{T}} \mathbf{t}\,\mathrm{d}\Gamma + \int_{\Gamma_{jk}^2} \mathbf{N}^{2\,\mathrm{T}} \mathbf{t}\,\mathrm{d}\Gamma + \int_{\Gamma_{ki}^2} \mathbf{N}^{2\,\mathrm{T}} \mathbf{t}\,\mathrm{d}\Gamma$$

$$= \int_0^2 \left(\mathbf{N}^2(x^{(2)}, y^{(2)} = 2)\right)^\mathrm{T} \begin{bmatrix} 0 \\ -6\left(1 - \frac{x^{(2)}}{2}\right) - 7.5\frac{x^{(2)}}{2} \end{bmatrix} \mathrm{d}x^{(2)}$$

$$= \begin{bmatrix} 0 \\ 0 \\ 0 \\ -7 \\ 0 \\ -6.5 \end{bmatrix}$$

5. Agregacja – Macierze Boole'a $\mathbb{B}$

- Element 1

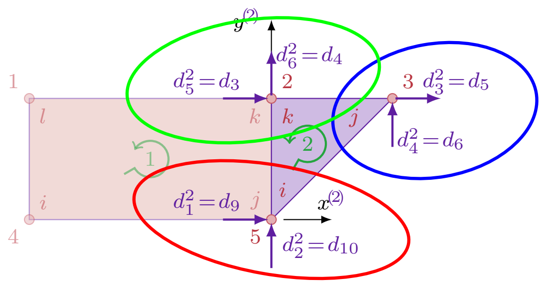


nr. lok.

$$\mathbb{B}^1 = \begin{bmatrix} 0&0&0&0&0&0&1&0&0&0 \\ 0&0&0&0&0&0&0&1&0&0 \\ 0&0&0&0&0&0&0&0&1&0 \\ 0&0&0&0&0&0&0&0&0&1 \\ 0&0&1&0&0&0&0&0&0&0 \\ 0&0&0&1&0&0&0&0&0&0 \\ 1&0&0&0&0&0&0&0&0&0 \\ 0&1&0&0&0&0&0&0&0&0 \end{bmatrix} \begin{matrix} (1) \\ (2) \\ (3) \\ (4) \\ (5) \\ (6) \\ (7) \\ (8) \end{matrix}$$

nr. glob. (1) (2) (3) (4) (5) (6) (7) (8) (9) (10)

- Element 2



nr. lok.

$$\mathbb{B}^2 = \begin{bmatrix} 0&0&0&0&0&0&0&0&1&0 \\ 0&0&0&0&0&0&0&0&0&1 \\ 0&0&0&0&1&0&0&0&0&0 \\ 0&0&0&0&0&1&0&0&0&0 \\ 0&0&1&0&0&0&0&0&0&0 \\ 0&0&0&1&0&0&0&0&0&0 \end{bmatrix} \begin{matrix} (1) \\ (2) \\ (3) \\ (4) \\ (5) \\ (6) \end{matrix}$$

nr. glob. (1) (2) (3) (4) (5) (6) (7) (8) (9) (10)

6. Agregacja - Macierz sztywności

$$\mathbf{K} = {\mathbb{B}^1}^{\mathrm{T}} \mathbf{K}^1 \mathbb{B}^1 + {\mathbb{B}^2}^{\mathrm{T}} \mathbf{K}^2 \mathbb{B}^2$$

$$\mathbf{K} = \begin{bmatrix} 16 & -6 & -1.6 & 1.2 & 0 & 0 & -6.4 & -1.2 & -8 & 6 \\ -6 & 28 & -1.2 & 10.4 & 0 & 0 & 1.2 & -24.4 & 6 & -14 \\ -1.6 & -1.2 & 42.4 & -6 & -19.2 & 7.2 & -8 & -6 & -13.6 & 6 \\ 1.2 & 10.4 & -6 & 54.4 & 4.8 & -7.2 & -6 & -14 & 6 & -43.6 \\ 0 & 0 & -19.2 & 4.8 & 19.2 & 0 & 0 & 0 & 0 & -4.8 \\ 0 & 0 & 7.2 & -7.2 & 0 & 7.2 & 0 & 0 & -7.2 & 0 \\ -6.4 & 1.2 & -8 & -6 & 0 & 0 & 16 & 6 & -1.6 & -1.2 \\ -1.2 & -24.4 & -6 & -14 & 0 & 0 & 6 & 28 & 1.2 & 10.4 \\ -8 & 6 & -13.6 & 6 & 0 & -7.2 & -1.6 & 1.2 & 23.2 & -6 \\ 6 & -14 & 6 & -43.6 & -4.8 & 0 & -1.2 & 10.4 & -6 & 47.2 \end{bmatrix} \cdot 10^5$$

7. Agregacja - Wektor obciążenia

$$\mathbf{p}_{\mathrm{b}} = {\mathbb{B}^1}^{\mathrm{T}} \mathbf{p}_{\mathrm{b}}^1 + {\mathbb{B}^2}^{\mathrm{T}} \mathbf{p}_{\mathrm{b}}^2, \qquad \mathbf{p} = \mathbf{0}$$

$$\mathbf{p}_{\mathrm{b}} = \begin{bmatrix} 0 \\ -8 \\ 0 \\ -16.5 \\ 0 \\ -7 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} r_7^1 = r_1 \\ r_8^1 = r_2 \\ 0 \\ 0 \\ 0 \\ 0 \\ r_1^1 = r_7 \\ r_2^1 = r_8 \\ 0 \\ 0 \end{bmatrix}$$

8. Układ równań MES: $\mathbf{Kd} = \mathbf{p} + \mathbf{p}_b$

$$\begin{bmatrix} 16 & -6 & -1.6 & 1.2 & 0 & 0 & -6.4 & -1.2 & -8 & 6 \\ -6 & 28 & -1.2 & 10.4 & 0 & 0 & 1.2 & -24.4 & 6 & -14 \\ -1.6 & -1.2 & 42.4 & -6 & -19.2 & 7.2 & -8 & -6 & -13.6 & 6 \\ 1.2 & 10.4 & -6 & 54.4 & 4.8 & -7.2 & -6 & -14 & 6 & -43.6 \\ 0 & 0 & -19.2 & 4.8 & 19.2 & 0 & 0 & 0 & 0 & -4.8 \\ 0 & 0 & 7.2 & -7.2 & 0 & 7.2 & 0 & 0 & -7.2 & 0 \\ -6.4 & 1.2 & -8 & -6 & 0 & 0 & 16 & 6 & -1.6 & -1.2 \\ -1.2 & -24.4 & -6 & -14 & 0 & 0 & 6 & 28 & 1.2 & 10.4 \\ -8 & 6 & -13.6 & 6 & 0 & -7.2 & -1.6 & 1.2 & 23.2 & -6 \\ 6 & -14 & 6 & -43.6 & -4.8 & 0 & -1.2 & 10.4 & -6 & 47.2 \end{bmatrix} \cdot 10^5 \begin{bmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \\ d_5 \\ d_6 \\ d_7 \\ d_8 \\ d_9 \\ d_{10} \end{bmatrix} = \begin{bmatrix} 0 \\ -8 \\ 0 \\ -16.5 \\ 0 \\ -7 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} r_1 \\ r_2 \\ 0 \\ 0 \\ 0 \\ 0 \\ r_7 \\ r_8 \\ 0 \\ 0 \end{bmatrix}$$

9. Uwzglęnienie warunków brzegowych

$$\begin{bmatrix} 16 & -6 & -1.6 & 1.2 & 0 & 0 & -6.4 & -1.2 & -8 & 6 \\ -6 & 28 & -1.2 & 10.4 & 0 & 0 & 1.2 & -24.4 & 6 & -14 \\ -1.6 & -1.2 & 42.4 & -6 & -19.2 & 7.2 & -8 & -6 & -13.6 & 6 \\ 1.2 & 10.4 & -6 & 54.4 & 4.8 & -7.2 & -6 & -14 & 6 & -43.6 \\ 0 & 0 & -19.2 & 4.8 & 19.2 & 0 & 0 & 0 & 0 & -4.8 \\ 0 & 0 & 7.2 & -7.2 & 0 & 7.2 & 0 & 0 & -7.2 & 0 \\ -6.4 & 1.2 & -8 & -6 & 0 & 0 & 16 & 6 & -1.6 & -1.2 \\ -1.2 & -24.4 & -6 & -14 & 0 & 0 & 6 & 28 & 1.2 & 10.4 \\ -8 & 6 & -13.6 & 6 & 0 & -7.2 & -1.6 & 1.2 & 23.2 & -6 \\ 6 & -14 & 6 & -43.6 & -4.8 & 0 & -1.2 & 10.4 & -6 & 47.2 \end{bmatrix} \cdot 10^5 \begin{bmatrix} 0 \\ 0 \\ d_3 \\ d_4 \\ d_5 \\ d_6 \\ 0 \\ 0 \\ d_9 \\ d_{10} \end{bmatrix} = \begin{bmatrix} 0 \\ -8 \\ 0 \\ -16.5 \\ 0 \\ -7 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} r_1 \\ r_2 \\ 0 \\ 0 \\ 0 \\ 0 \\ r_7 \\ r_8 \\ 0 \\ 0 \end{bmatrix}$$

Rozwiązanie:

$$\mathbf{d} = \{0 \ 0 \ 3.881 \ \text{-}11.03 \ 3.949 \ \text{-}19.62 \ 0 \ 0 \ \text{-}3.744 \ \text{-}10.75\} \cdot 10^{-5} \text{ m}$$

$$\mathbf{r} = \{\text{-}54 \ 16.744 \ 0 \ 0 \ 0 \ 0 \ 54 \ 14.756 \ 0 \ 0\} \text{ kN}$$

10. Powrót do elementu

- Element 1

$$\mathbf{d}^1 = \mathbb{B}^1 \mathbf{d} = \{0 \ 0 \ \text{-}3.744 \ \text{-}10.75 \ 3.881 \ \text{-}11.03 \ 0 \ 0\} \cdot 10^{-5}$$

$$\mathbf{e}^1 = \mathbf{B}^1\mathbf{d}^1$$

$$\mathbf{e}^1 = \begin{bmatrix} 0.953y - 0.936 \\ -0.034x \\ 0.953x - 0.034y - 2.688 \end{bmatrix} \cdot 10^{-5}, \qquad \mathbf{e}^1(2,1) = \begin{bmatrix} 1.708 \\ 6.831 \\ -81.600 \end{bmatrix} \cdot 10^{-7}$$

$$\mathbf{s}^1 = \mathbf{D}\mathbf{e}^1$$

$$\mathbf{s}^1 = \begin{bmatrix} 182.976y - 179.712 - 1.632x \\ 45.744y - 44.928 - 6.528x \\ 68.616x - 2.448y - 193.536 \end{bmatrix}, \qquad \mathbf{s}^1(2,1) = \begin{bmatrix} 0 \\ -12.297 \\ -58.750 \end{bmatrix} \text{ kPa}$$

- Element 2

$$\mathbf{d}^2 = \mathbb{B}^2\mathbf{d} = \{\text{-3.744 -10.75 3.949 -19.62 3.881 -11.03}\} \cdot 10^{-5}$$

$$\mathbf{e}^2 = \mathbf{B}^2\mathbf{d}^2$$

$$\mathbf{e}^2 = \begin{bmatrix} 3.416 \\ 13.660 \\ -48.610 \end{bmatrix} \cdot 10^{-7}$$

$$\mathbf{s}^2 = \mathbf{D}\mathbf{e}^2$$

$$\mathbf{s}^2 = \begin{bmatrix} 0 \\ -24.593 \\ -35.000 \end{bmatrix} \text{ kPa}$$

11. Wyznaczenie wartości przemieszczeń w wybranych punktach

- Element 1 – punkt ($x^{(1)} = 2, y^{(1)} = 1$)

$$\mathbf{u}^1(x^{(1)}, y^{(1)}) = \mathbf{N}^1(x^{(1)}, y^{(1)})\mathbf{d}^1$$

$$\mathbf{u}^1(2,1) = \begin{bmatrix} \frac{1}{4} & 0 & \frac{1}{4} & 0 & \frac{1}{4} & 0 & \frac{1}{4} & 0 \\ 0 & \frac{1}{4} & 0 & \frac{1}{4} & 0 & \frac{1}{4} & 0 & \frac{1}{4} \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 0 \\ -3.744 \\ -10.75 \\ 3.881 \\ -11.03 \\ 0 \\ 0 \end{bmatrix} \cdot 10^{-5} = \begin{bmatrix} 0.0343 \\ -5.4450 \end{bmatrix} \cdot 10^{-5} \text{ [m]}$$

- Element 2 – punkt ($x^{(2)} = 0.5, y^{(2)} = 1.5$)

$$\mathbf{u}^2(x^{(2)}, y^{(2)}) = \mathbf{N}^2(x^{(2)}, y^{(2)})\mathbf{d}^2$$

$$\mathbf{u}^2(0.5,1.5) = \begin{bmatrix} \frac{1}{4} & 0 & \frac{1}{4} & 0 & \frac{1}{2} & 0 \\ 0 & \frac{1}{4} & 0 & \frac{1}{4} & 0 & \frac{1}{2} \end{bmatrix} \cdot \begin{bmatrix} -3.744 \\ -10.75 \\ 3.949 \\ -19.62 \\ 3.881 \\ -11.03 \end{bmatrix} \cdot 10^{-5} = \begin{bmatrix} 1.9916 \\ -13.1065 \end{bmatrix} \cdot 10^{-5} \text{ [m]}$$