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OPTIMAL LAYOUTS OF A TWO-PHASE ISOTROPIC MATERIAL IN THIN ELASTIC PLATES

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Abstract. *The paper concerns the problem of minimization of the total compliance of the two-phase plates. These phases are characterized by two different thicknesses:* h_1, h_2, $h_2 > h_1$. *The relaxation of the problem follows the paper by Gibiansky and Cherkaev (report No 914, Fiz. Tekhn. Inst. im. A. Ioffe, AN SSSR, Leningrad 1984). Thus the composite domains are admitted where the microstructure is that of 2nd rank ribbed plates with mutually orthogonal stiffening. Distribution of the three design variables:* $\omega_1(x)$, $\theta_1(x)$, $\phi(x)$ *are found iteratively by using the updating scheme of M. Bendsøe. The layouts obtained are characterized by some properties that can be predicted by qualitative optimization methods.*

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1 Introduction

The paper deals with minimizing the total compliance of thin elastic plates. If a distributed transverse loading $p = p(x)$ causes the deflection field $w = w(x)$ the compliance means simply

$$J = \int_\Omega p(x)w(x)\,dx\ ; \tag{1}$$

here $x = (x_1, x_2) \in \Omega$, Ω being the plate middle plane. Thus minimization of J is equivalent to maximization of an overall stiffness of the plate.

Let the plate stiffness tensor be non-homogeneous: $\boldsymbol{D} = \boldsymbol{D}(x)$, $\boldsymbol{D} = (D^{\alpha\beta\lambda\mu})$. The deflection field w entering (1) satisfies the variational equilibrium equation

$$\int_\Omega v_{,\alpha\beta} D^{\alpha\beta\lambda\mu}(x) w_{,\lambda\mu}\,dx = \int_\Omega pv\,dx \tag{2}$$

for all v being kinematically admissible. Assume that the material of the plate is given as isotropic, with moduli E and ν. Let the plate thickness h assume two values: h_2 and h_1, $h_2 > h_1 > 0$. We write $h(x) \in \{h_1, h_2\}$ or $h(x) = h_1\chi_1(x) + h_2\chi_2(x)$, $\chi_2(x) = 1 - \chi_1(x)$ and $\chi_1(x)$ equals 1 in Ω_1, where $h = h_1$ and $\chi_1 = 0$ in $\Omega_2 = \Omega \backslash \Omega_1$. Note that χ_α is a characteristic function of the domain Ω_α.

We consider a family of plates of a given volume

$$V = h_1 \int_\Omega \chi_1\,dx + h_2 \int_\Omega \chi_2\,dx \tag{3}$$

One of the oldest and still challenging optimum design problems reads:

(P) | find χ_1 such that J assumes minimum under the condition of the volume being fixed, see (3). The field w entering (1) satisfies the equation (2) in which the relation $\boldsymbol{D}(h)$ corresponds to the thin plate theory.

A quixotic history of attempts of solving the above problem directly has eventually ended up in 1984, when Gibiansky and Cherkaev [1] proposed a correct relaxation of the problem. It has turned out that the problem (P) should be replaced with a new one $(\widetilde{P})$ in which: $\chi_\alpha(x)$ are replaced with $m_\alpha(x)$, $\boldsymbol{D}(x)$ is replaced with $\widetilde{\boldsymbol{D}}(x)$ where $m_\alpha(x)$ are continuous functions defined in Ω and assuming values in $[0, 1]$, while $\widetilde{\boldsymbol{D}}$ corresponds to a thin composite plate constructed by two subsequent and orthogonal layerings, in the sense of the homogenization theory of plates, see Lewiński and Telega [2].

The optimum designs in the relaxed sense $(\widetilde{P})$ has been a subject of the papers by Gibiansky and Cherkaev [1], Diaz et al. [3], Krog and Olhoff [4], Lewiński and Othman [5], Othman [6], Kolanek and Lewiński [7], Olhoff et al. [8]. Since only selected optimization problems for thin plates are presented in this literature it is thought appropriate to put forward here some relaxed solutions for thin simply supported and clamped quadratic plates, subjected to a uniform loading and discuss their properties. The characteristic features of the relaxed solutions are:

i) the optimal plate is isotropic and non-homogeneous but can, paradoxically, be treated as orthotropic in some subdomains;

ii) there are three types of those subdomains; they are determined by the value of the non-dimensional invariant of the moment state: $||\,\text{deviator}\,(\boldsymbol{M})||/|tr(\boldsymbol{M})|$, the norm $||\cdot||$ being defined as Euclidean;

iii) in two of those subdomains the principal directions of $\boldsymbol{M}$ determine the direction of ribs at the microscale. In the third subdomain one observes a deviation between these directions;

iv) in all subdomains the directions of principal moments and principal strains (changes of curvature) coincide.

The aim of the paper is to check these properties by numerical computation and to discuss the morphology of the mentioned subdomains, for the variable initial data: h_1/h_2 and V.

2 Relaxed formulation

For isotropy the bending stiffness tensor is represented by

$$\boldsymbol{D} = 2k\boldsymbol{I}_1 + 2\mu\boldsymbol{I}_2\,, \tag{4}$$

where

$$k = \frac{Eh^3}{24(1-\nu)}\,, \qquad \mu = \frac{Eh^3}{24(1+\nu)} \tag{5}$$

and the components of tensors $\boldsymbol{I}_\alpha$ are given by

$$\boldsymbol{I}_1 = \left(\frac{1}{2}\delta^{\alpha\beta}\delta^{\lambda\mu}\right) \qquad \boldsymbol{I}_2 = \left(\frac{1}{2}\left(\delta^{\alpha\lambda}\delta^{\mu\beta} + \delta^{\alpha\mu}\delta^{\beta\lambda} - \delta^{\alpha\beta}\delta^{\lambda\mu}\right)\right) \tag{6}$$

The bending stiffness tensor is represented by $\boldsymbol{D}_1$ in Ω_1 and $\boldsymbol{D}_2$ in Ω_2, where

$$\boldsymbol{D}_\alpha = 2k_\alpha\boldsymbol{I}_1 + 2\mu_\alpha\boldsymbol{I}_2 \tag{7}$$

with

$$k_\alpha = \frac{E(h_\alpha)^3}{24(1-\nu)}\,, \qquad \mu_\alpha = \frac{E(h_\alpha)^3}{24(1+\nu)} \tag{8}$$

The condition $h_2 > h_1$ implies

$$k_2 > k_1\,, \qquad \mu_2 > \mu_1 \tag{9}$$

and $\boldsymbol{D}_2 - \boldsymbol{D}_1$ is positive definite, hence invertible.

The homogenized tensor $\widetilde{\boldsymbol{D}}$ is constructed in two steps. First we construct a ribbed plate along the x_1 direction, with area fractions θ_1 and $\theta_2 = 1 - \theta_1$. Then the homogenized material thus obtained is mixed with the stronger one to form ribs along x_1, with area fractions $\omega_1, \omega_2 = 1-\omega_1$ along x_2. This theoretical construction is explained in Lewiński and Telega [2, Sec. 24.2] in detail, hence it is sufficient to recall the final result. The non-zero components of $\widetilde{\boldsymbol{D}}$ referred to the orthonormal basis are

$$\widetilde{D}^{1111} = \frac{1}{2}\left(\widetilde{D}^{11} + 2\widetilde{D}^{12} + \widetilde{D}^{22}\right), \qquad \widetilde{D}^{2222} = \frac{1}{2}\left(\widetilde{D}^{11} - 2\widetilde{D}^{12} + \widetilde{D}^{22}\right),$$
$$\widetilde{D}^{1122} = \frac{1}{2}\left(\widetilde{D}^{11} - \widetilde{D}^{22}\right), \qquad \widetilde{D}^{1212} = \frac{1}{2}\widetilde{D}^{33}, \tag{10}$$
$$\widetilde{D}^{2211} = \widetilde{D}^{1122}, \qquad \widetilde{D}^{2121} = \widetilde{D}^{1221} = \widetilde{D}^{2112} = \widetilde{D}^{1212}$$

where

$$\frac{1}{2}\widetilde{D}^{11} = k_2 - \frac{m_1(k_2 + [\mu]_m)\Delta k(k_2 + \mu_2)}{\eta(\theta_1, m_1)}$$
$$\frac{1}{2}\widetilde{D}^{12} = \frac{m_1(2\theta_1 - 1 - m_1)\Delta k \Delta\mu(k_2 + \mu_2)}{\eta(\theta_1, m_1)}$$
$$\frac{1}{2}\widetilde{D}^{22} = \mu_2 - \frac{m_1(\mu_2 + [k]_m)\Delta\mu(k_2 + \mu_2)}{\eta(\theta_1, m_1)} \tag{11}$$
$$\frac{1}{2}\widetilde{D}^{33} = \langle\mu\rangle_m, \qquad \widetilde{D}^{\alpha 3} = \widetilde{D}^{3\alpha} = 0$$

The following notation is introduced:

$$\eta(\theta_1, m_1) = 4(1-\theta_1)(\theta_1 - m_1)\Delta\mu\Delta k + (k_2 + \mu_2)[k + \mu]_m,$$
$$[g]_m = m_1 g_2 + m_2 g_1, \qquad \langle g\rangle_m = m_1 g_1 + m_2 g_2, \qquad \Delta g = |g_2 - g_1|, \tag{12}$$
$$\{g\}_m = (m_1 g_1^{-1} + m_2 g_2^{-1})^{-1}$$

for $g \in \{\mu, k\}$. Moreover,

$$m_2 = \omega_2 + \omega_1\theta_2, \qquad m_1 = \theta_1\omega_1 \tag{13}$$

and $m_1 + m_2 = 1$.

If the microstructure of properties (10) is rotated by an angle ϕ with respect to the original coordinate system, then the stiffnesses $\widetilde{D}^{ij}_\phi$ referred to the original orthonormal basis are determined by the transformation rules:

$$\widetilde{D}^{11}_\phi = \widetilde{D}^{11}, \qquad \widetilde{D}^{12}_\phi = \cos 2\phi\,\widetilde{D}^{12}, \qquad \widetilde{D}^{13}_\phi = \sin 2\phi\,\widetilde{D}^{12},$$
$$\widetilde{D}^{22}_\phi = \cos^2 2\phi\,\widetilde{D}^{22} + \sin^2 2\phi\,\widetilde{D}^{33}, \qquad \widetilde{D}^{23}_\phi = \sin 2\phi \cos 2\phi\left(\widetilde{D}^{22} - \widetilde{D}^{33}\right), \tag{14}$$
$$\widetilde{D}^{33}_\phi = \sin^2 2\phi\,\widetilde{D}^{22} + \cos^2 2\phi\,\widetilde{D}^{33}, \qquad \widetilde{D}^{ij}_\phi = \widetilde{D}^{ji}_\phi$$

and the components $\widetilde{D}^{\alpha\beta\lambda\mu}_\phi$ are given by (10) with $\widetilde{D}^{ij}$ replaced by $\widetilde{D}^{ij}_\phi$. We shall use the following notation

$$\widetilde{D}^{\alpha\beta\lambda\mu}_\phi = \widetilde{D}^{\alpha\beta\lambda\mu}(\theta_2, \omega_2, \phi) \tag{15}$$

to stress that $\widetilde{\boldsymbol{D}}_\phi$ is determined by two area fractions: θ_2 and ω_2, and an angle ϕ; their range is:

$$0 \le \theta_2 \le 1, \qquad 0 \le \omega_2 \le 1, \qquad 0 \le \phi \le 2\pi \tag{16}$$

Just the functions $\theta_2(x)$, $\omega_2(x)$ and $\phi(x)$ are design variables of the relaxed problem $(\widetilde{P})$. This problem consists in minimizing the functional J, see (1), where w solves the variational equation (2) with $\boldsymbol{D}$ replaced by $\widetilde{\boldsymbol{D}}(\theta_2, \omega_2, \phi)$, and the isoperimetric condition (3) is replaced by

$$\int_\Omega [h_1 m_1(x) + h_2 m_2(x)] \; dx = V \tag{17}$$

The fields m_1, m_2 are expressed by Ω_2 and θ_2 by (13).

For special values of ω_2 and θ_2 the relaxed formulation encompasses the initial one, but the optimal solution usually admits a region where $0 < \theta_2 < 1$, $0 < \omega_2 < 1$ and there the plate has an undetermined thickness.

3 Direct numerical solution of the relaxed problem

3.1 Necessary conditions of optimality

We shall follow the algorithm of Bendsøe [9]. It is helpful to recall it here.

Let us introduce the bilinear form

$$a(w,v) = \int_\Omega v_{,\alpha\beta} \widetilde{D}^{\alpha\beta\lambda\mu}(\theta_2, \omega_2, \phi) w_{,\lambda\mu} \, dx \tag{18}$$

and the linear form

$$f(v) = \int_\Omega pv \, dx \tag{19}$$

The conditions (16) are expressed with using slack variables $\alpha_1, \ldots, \alpha_4$ as follows

$$\begin{aligned} -\theta_2 + (\alpha_1)^2 = 0 \,, \qquad & -1 + \theta_2 + (\alpha_2)^2 = 0 \,, \\ -\omega_2 + (\alpha_3)^2 = 0 \,, \qquad & -1 + \omega_2 + (\alpha_4)^2 = 0 \end{aligned} \tag{20}$$

Note that $J = f(w)$ and w satisfies: $a(w,v) = f(v)$ for all kinematically admissible fields v. We define the Lagrangian function:

$$\begin{aligned} L = f(w) + [f(v) - a(w,v)] + \Lambda \left\{ \int_\Omega [(h_2 - h_1)(\theta_2 + \omega_2 - \omega_2\theta_2) + h_1] \, dx - V \right\} + \\ + \int_\Omega [\lambda_1(-\theta_2 + (\alpha_1)^2) + \lambda_2(-1 + \theta_2 + (\alpha_2)^2) + \lambda_3(-\omega_2 + (\alpha_3)^2) + \lambda_4(-1 + \omega_2 + (\alpha_4)^2)] \, dx \end{aligned} \tag{21}$$

Note that v plays simultaneously two roles: of a trial field and of the Lagrangian multiplier for the equilibrium equation:

$$a(w,v) = f(v) \tag{22}$$

The stationary condition $\delta L = 0$ with respect to w, ϕ, θ_2, ω_2, α_i imply

$$f(\delta w) - a(v, \delta w) = 0 \tag{23}$$

$$\frac{\partial L}{\partial \phi} = 0 \;, \tag{24}$$

$$\frac{\partial L}{\partial \theta_2} = 0 \;, \tag{25}$$

$$\frac{\partial L}{\partial \omega_2} = 0 \;, \tag{26}$$

$$\frac{\partial L}{\partial \alpha_i} = 0 \tag{27}$$

The equation (23) implies $v = w$, where w solves (22). The equation (24) gives a pointwise condition

$$\frac{\partial}{\partial \phi}\left[\kappa_{\alpha\beta}\widetilde{D}^{\alpha\beta\lambda\mu}(\theta_2, \omega_2, \phi)\kappa_{\lambda\mu}\right] = 0 \tag{28}$$

where $\kappa_{\alpha\beta}(v) = -v_{,\alpha\beta}$. The roots of this equation can be found analytically, see Fedorov and Cherkaev [10], Banichuk [11], Pedersen [12]:

$$\begin{gathered} \phi_1 = \phi_\kappa \;, \qquad \phi_{2,3} = \phi_\kappa \pm \frac{\pi}{2} \;, \\ \phi_{4,5} = \phi_\kappa \pm \frac{1}{2}\arccos(-\beta_\kappa) \;, \end{gathered} \tag{29}$$

where

$$\beta_\kappa = \frac{\widetilde{D}^{12}}{\widetilde{D}^{22} - \widetilde{D}^{33}} \frac{tr\boldsymbol{\kappa}}{\sqrt{(tr\boldsymbol{\kappa})^2 - 4\det\boldsymbol{\kappa}}} \tag{30}$$

and ϕ_κ represents an angle between the first principal direction of $\boldsymbol{\kappa}$ and the axis x_1.

The equations (25), (26) imply the following conditions

$$\frac{\partial}{\partial \theta_2}\left[\kappa_{\alpha\beta}(w)\widetilde{D}^{\alpha\beta\lambda\mu}(\theta_2, \omega_2, \phi)\kappa_{\lambda\mu}(w)\right] = \Lambda(1-\omega_2)(h_2 - h_1) - \lambda_1 + \lambda_2 \;, \tag{31}$$

$$\frac{\partial}{\partial \omega_2}\left[\kappa_{\alpha\beta}(w)\widetilde{D}^{\alpha\beta\lambda\mu}(\theta_2, \omega_2, \phi)\kappa_{\lambda\mu}(w)\right] = \Lambda(1-\theta_2)(h_2 - h_1) - \lambda_3 + \lambda_4 \tag{32}$$

Let us define the auxiliary quantities

$$Q = \frac{\kappa_{\alpha\beta}(w)\dfrac{\partial \widetilde{D}^{\alpha\beta\lambda\mu}}{\partial \theta_2}\kappa_{\lambda\mu}(w)}{\Lambda(1-\omega_2)(h_2 - h_1)} \;, \qquad P = \frac{\kappa_{\alpha\beta}(w)\dfrac{\partial \widetilde{D}^{\alpha\beta\lambda\mu}}{\partial \omega_2}\kappa_{\lambda\mu}(w)}{\Lambda(1-\theta_2)(h_2 - h_1)} \tag{33}$$

If $\lambda_1 = 0$ and $\theta_2 \neq 1$, $\omega_2 \neq 1$, the conditions (31), (32) assume the form

$$Q = 1 \;, \qquad P = 1 \tag{34}$$

The stationary conditions with respect to α_i are $\alpha_i \lambda_i = 0$ (do not sum over i)

The condition $\dfrac{\partial^2 L}{\partial(\alpha_i)^2} \geq 0$ implies $\lambda_i \geq 0$.

3.2 The computational algorithm

The equilibrium equation (22) is solved approximately by the three-node triangular plate bending finite element method (element DKT), as developed in Batoz et al. [13] and Kikuchi [14]. The analysis and optimization program has been written in C++ and graphical procedures have been written in Java. The numerical solution to the problem $(\widetilde{P})$ is constructed iteratively, the main steps being similar to those applied in Lewiński and Othman [5] and in Othman [6].

Step 0. Fix the data: E, ν, h_1, h_2, Ω, $p = p(x)$, V and the boundary conditions on $\partial\Omega$. Discretize Ω into the elements Ω_e, $e = 1, 2, \ldots, Ne$; assign $(\theta_2^e, \omega_2^e, \phi^e)$ for each Ω_e. We apply the simplest possible approximation

$$f(x) = \sum_{e=1}^{Ne} f^e \chi_{\Omega_e}(x) \ , \qquad f \in \{\theta_2, \omega_2, \phi\} \tag{35}$$

Step 1. Compute $\widetilde{D}^{\alpha\beta\lambda\mu}(\theta_2^e, \omega_2^e, \phi^e)$ and $\widetilde{D}^{ij}(\theta_2^e, \omega_2^e, \phi^e)$.

Find w representing the deflection field satisfying (22).

If the step 1 is executed first time, compute $J = f(w)$ by (19) and initialize $J_{old} = J$. In opposite case, skip this command.

Compute $\kappa_{\alpha\beta}^e(w)$

Compute the roots ϕ_i^e, $i = 1, \ldots, 5$ by (29)

Find the energy density for each element;

$$U_i^e = \frac{1}{2}\kappa_{\alpha\beta}^e(w)\widetilde{D}^{\alpha\beta\lambda\mu}(\theta_2^e, \omega_2^e, \phi_i^e)\kappa_{\lambda\mu}^e(w)$$

Choose this root ϕ_j^e for which U_j^e is the greatest for each element independently.

Step 2. Fix an initial value of the multiplier Λ

Step 3. For each element compute analytically the derivatives:

$$\frac{\partial\widetilde{D}^{\alpha\beta\lambda\mu}(\theta_2^e, \omega_2^e, \phi^e)}{\partial\theta_2} \ , \qquad \frac{\partial\widetilde{D}^{\alpha\beta\lambda\mu}(\theta_2^e, \omega_2^e, \phi^e)}{\partial\omega_2} \ ,$$

and Q^e, P^e by (33).

Fix $\zeta \in [0, 1]$ called a move limit. Choose a non-negative η, say $\eta = 0.75$, a damped factor. Update θ_2 by the scheme of Bendsøe [9]

$$\theta_2^e = \begin{cases} \max\{(1-\zeta)\theta_2^e, 0\} & \text{if (i)} \\ \theta_2^e(Q^e)^\eta & \text{if (ii)} \\ \min\{(1+\zeta)\theta_2^e, 1\} & \text{if (iii)} \end{cases} \tag{36}$$

The conditions (i)-(iii) are defined as follows

(i) $\theta_2^e(Q^e)^\eta \le \max\{(1-\zeta)\theta_2^e, 0\}$

(ii) $\max\{(1-\zeta)\theta_2^e, 0\} \le \theta_2^e(Q^e)^\eta \le \min\{(1+\zeta)\theta_2^e, 1\}$

(iii) $\min\{(1+\zeta)\theta_2^e, 1\} \le \theta_2^e(Q^e)^\eta$

To update ω_2 a similar scheme is used, with Q replaced by P. Thus we find ω_2. A finite element will be called passive if (ii) holds and active in other cases.

In the case when "old" θ_2^e assumes the values 0 or 1 or ω_2^e assumes these values the new values of θ_2^e and ω_2^e are not changed.

Compute the volume

$$V_\Lambda = \sum_{e-1}^{Ne} \int_{\Omega_e} \left[(h_2-h_1)(\theta_2^e + \omega_2^e - \theta_2^e\omega_2^e) + h_1\right]\, dx$$

As noted by Othman [6], if $V_\Lambda > V$ one should increase Λ and go to Step 3. If $V_\Lambda < V$ then decrease Λ and go to Step 3. If $V_\Lambda \approx V$ then, by a standard routine, compute the compliance J and initialize $J_{new} = J$.

If $J_{new} \approx J_{old}$ then STOP. In opposite case set $J_{old} = J_{new}$ and go to Step 1.

4 A smart material formulation

By using the duality methods presented in Gibiansky and Cherkaev [1], Allaire et al. [15], Allaire and Kohn [16], Lewiński and Telega [2] and Telega and Lewiński [17] one can reformulate the problem $(\widetilde{P})$ considered in Sec. 2 to the following minimization problem

$$(\widehat{P}) \min_{0\le m_2\le 1} \min_{\boldsymbol{M}\in S} \left\{ \int_\Omega 2W^*(\boldsymbol{M}(x), m_2(x))\, dx \mid \text{Eq. (17) holds} \right\}$$

Here S represents the set of statically admissible moments $\boldsymbol{M} = (M^{\alpha\beta})$ and the potential W^* is given by

$$2W^* = \begin{cases} \frac{1}{2}(I(\boldsymbol{M}))^2 H(\xi_M) & \text{if} \quad I(\boldsymbol{M}) \ne 0 \\ \frac{1}{2}\{L\}_m(I\!I(\boldsymbol{M}))^2 & \text{if} \quad I(\boldsymbol{M}) = 0 \end{cases} \tag{37}$$

where

$$\xi_M = \frac{I\!I(\boldsymbol{M})}{|I(\boldsymbol{M})|},\ I(\boldsymbol{M}) = \frac{1}{\sqrt{2}} tr\boldsymbol{M}\ , \qquad I\!I(\boldsymbol{M}) = \frac{1}{\sqrt{2}}\left[(tr\boldsymbol{M})^2 - 4\det \boldsymbol{M}\right]^{1/2} \tag{38}$$

The function $H(\xi)$ is defined by

$$\begin{cases} H_1(\xi) & \text{if} \quad \xi \ge \xi_1 \\ \widetilde{H}(\xi) & \text{if} \quad \xi \in [\xi_2, \xi_1] \\ H_2(\xi) & \text{if} \quad \xi \in [0, \xi_2] \end{cases} \tag{39}$$

$$H_\alpha(\xi) = a_\alpha + c_\alpha \xi^2 \ ,$$
$$\widetilde{H}(\xi) = H_\alpha(\xi) + d_\alpha(\xi - \xi_\alpha)^2 \ , \quad \alpha = 1, 2. \tag{40}$$

The parameters involved in (39)-(40) depend on $K_\alpha = (k_\alpha)^{-1}$, $L_\alpha = (\mu_\alpha)^{-1}$ and m_α:

$$\begin{gathered}
\xi_1 = \frac{\Delta K [L]_m}{\Delta L [K]_m} \ , \qquad \xi_2 = \frac{m_2 \Delta K}{L_2 + [K]_m} \ , \\
a_1 = \{K\}_m \ , \qquad a_2 = \frac{K_1 K_2 + L_2 \langle K \rangle_m}{L_2 + [K]_m} \ , \\
c_1 = \{L\}_m \ , \qquad c_2 = L_2 \\
d_1 = \frac{m_1 m_2 (\Delta L)^2 [K]_m}{[K+L]_m [L]_m} \ , \qquad d_2 = \frac{m_1 \Delta L (L_2 + [K]_m)}{[K+L]_m}
\end{gathered} \tag{41}$$

The operation $\{\cdot\}$ is explained in (12).

Note that H_α are stiched smoothly with $\widetilde{H}$ at $\xi_M = \xi_\alpha$. Hence the inverse constitutive relations: $\kappa_{\alpha\beta} = \dfrac{\partial W^*}{\partial M^{\alpha\beta}}$ are continuous.

Let us compare the formulations $(\widetilde{P})$ and $(\widehat{P})$. In $(\widehat{P})$ the function m_2 is the only design variable, in contrast to three design variables of the problem $(\widetilde{P})$. However, the static problem built in into $(\widehat{P})$ is nonlinear, which makes both the formulations equally difficult. It has turned out that the formulation $(\widehat{P})$ is well suited for one-dimensional problems of circular plates, see Kolanek and Lewiński [7]. The method is sufficiently flexible to consider the limit case of $h_1 = 0$, which corresponds to the shape optimization problem.

The result (37), (39) is essential to understand both the relaxation formulations $(\widetilde{P})$ and $(\widehat{P})$. We note that the plate domain is divided into 5 different subdomains, where, see Gibiansky and Cherkaev [1]:

a) $m_2 = 0$ or the plate is homogeneous and isotropic of constant thickness $h = h_1$

b) $m_2 = 1$, as above, but $h = h_2$

c) $\xi_M > \xi_1$ the microstructure is of first rank and ϕ is given by ϕ_4 or ϕ_5

d) $\xi_M \in [\xi_2, \xi_1]$; the microstructure is of first rank; $\phi = \phi_1$, ϕ_2 or ϕ_3;

e) $\xi_M \in [0, \xi_2]$; the microstructure is of second rank; $\phi = \phi_1$, ϕ_2 or ϕ_3;

The formulation $(\widehat{P})$ can be inverted to its primal form involving the displacement field w as the main unknown, see Lipton [18] and Lewiński and Telega [2]. Then the regimes (c), (d), (e) assume the form expressed in terms of strains:

c) $\xi_\kappa \geq \breve{\xi}_1$

d) $\xi_\kappa \in [\breve{\xi}_2, \breve{\xi}_1]$

e) $\xi_\kappa \in [0, \breve{\xi}_2]$,

where

$$\xi_\kappa = \frac{\sqrt{(tr\boldsymbol{\kappa})^2 - 4 \det \boldsymbol{\kappa}}}{|tr\boldsymbol{\kappa}|} \tag{42}$$

is a non-dimensional invariant characteristics of the strain field $\boldsymbol{\kappa}$.

Moreover,

$$\breve{\xi}_1 = \frac{\Delta k}{\Delta \mu}\,, \qquad \breve{\xi}_2 = \frac{m_2 \Delta k}{\mu_2 + [k]_m} \tag{43}$$

5 Optimal layouts of square plates subjected to a uniform loading

The numerical solutions presented here concern the square plates of side lengths 1.0 m by 1.0 m. The thicknesses are assumed as: $h_1 = 0.01\,\mathrm{m}$ and $h_2 = 0.02\,\mathrm{m}$. The volume is taken as $V = 0.001875\,\mathrm{m}^3$. The elastic moduli are: $E = 2.1 \cdot 10^8\,\mathrm{kN/m}^2$, $\nu = 0.3$. The density of a uniform dead loading applied equals: $p = 1.0\,\mathrm{kN/m}^2$.

Two cases of the homogeneous boundary conditions are considered: a clamped support along all the edges or a fully supported edge. Thus in all cases it is sufficient to consider 1/8 of the plate or confine consideration to the right- angled triangle of legs of length 0.5 m. This triangle domain is divided into triangular DKT finite elements, with 36 elements along a leg. The compliance of a homogeneous plate of a constant average thickness, having a given volume V is denoted by J_0. The ratio J/J_0 is called a relative compliance and its deviation from unity represents the advantage we have by redistribution of nonhomogeneities.

The optimal layout of the density of the second plate material: $m_2 = \omega_2 + (1 - \omega_2)\theta_2$ for the clamped plate of dimensions and elastic characteristics given above is presented in Fig. 1. Here the relative compliance equals: $J/J_0 = 0.66$. The whole range of the values of m_2 from 0 to 1 is represented by the grey scale- from white ($m_2 = 0$) to black ($m_2 = 1$).The strongest material, where the thickness equals h_2, finds its place along the sides. The corners are just the places where the weaker characteristics prevail. It turns out that the smaller value of the thickness is nowhere attained, since the corners and other intermediate regions are occupied by the composite materials described by the regimes (c)-(e), see Fig. 2. The domains of regimes are depicted by the grades of grey:

the light-grey regions correspond to the regime (c),

the grey regions characterize the regime (d) and

the dark-grey regions correspond to the regime (e).

Note that the regime (d) separates the regimes (c) and (e).

A different layout of m_2 is observed for simply supported plates, cf. Fig. 3. Here the central part is, as before, occupied by the 2nd rank ribbed plate but this domain is now almost circular. The first rank ribbed material (c) finds its place in the corners, see Fig. 4. The relative compliance is now $J/J_0 = 0.81$. Thus the advantage of optimization is smaller than in the case of clamped plates.

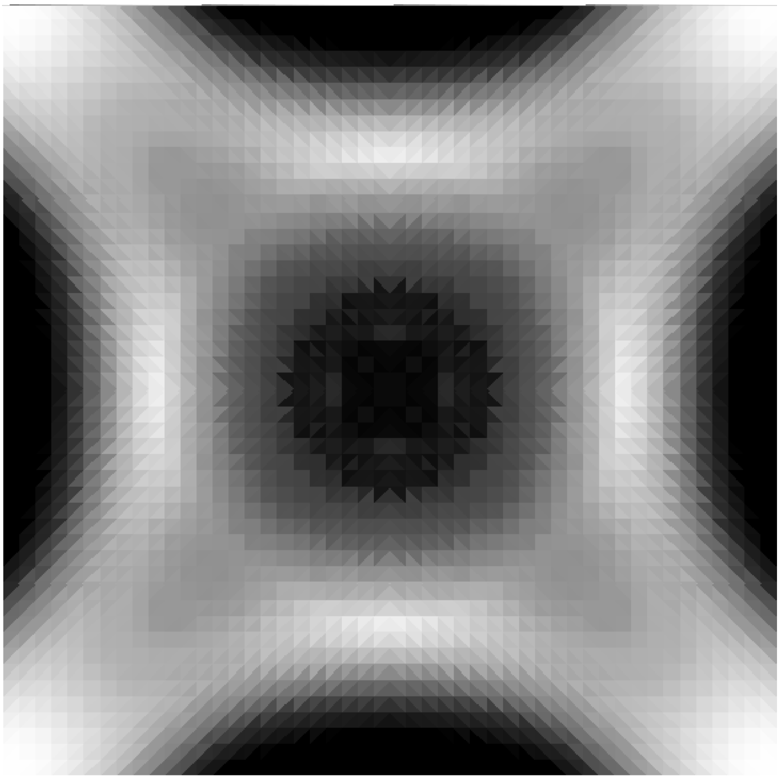

Fig. 1. Clamped plate. Distribution of m_2

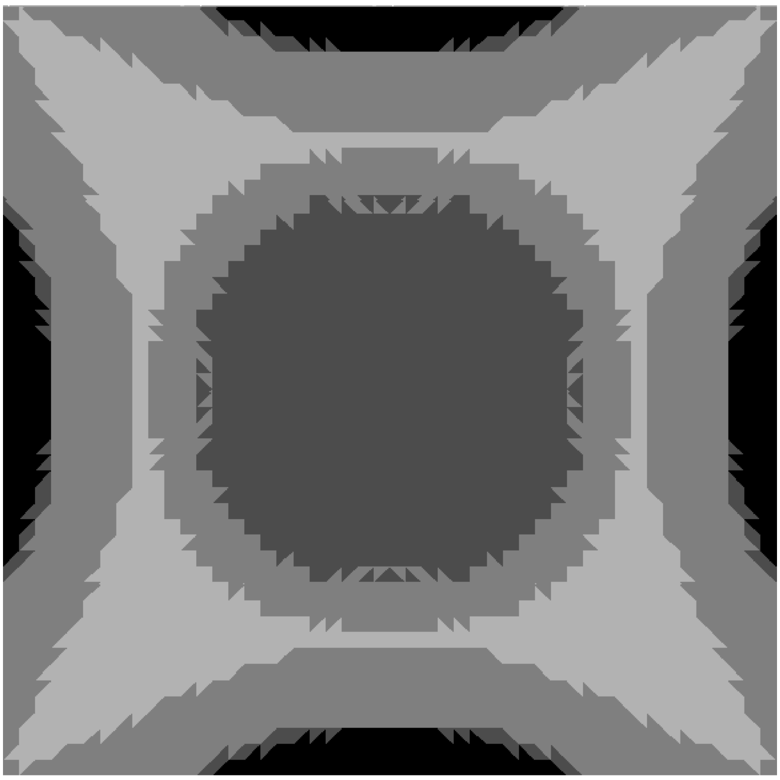

Fig. 2. Clamped plate. Layout of regimes: (b)-(e)

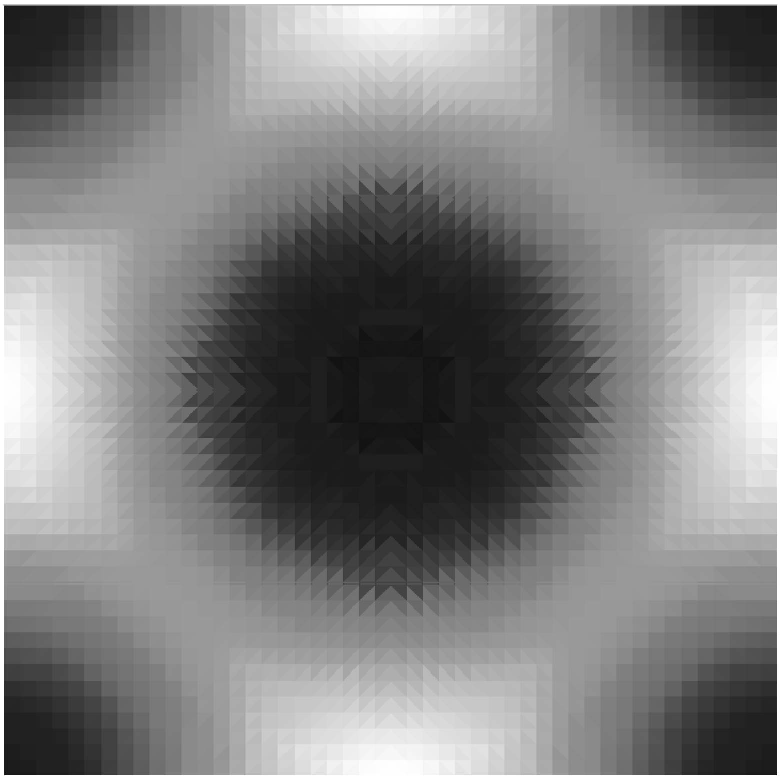

Fig. 3. Simply supported plate. Distribution of m_2

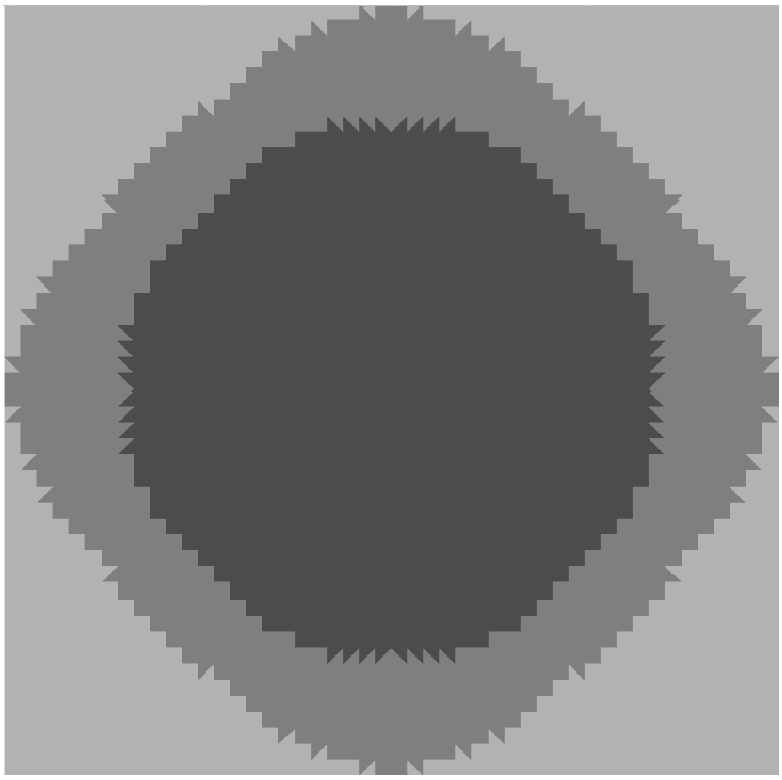

Fig. 4. Simply supported plate. Layout of regimes: (c)-(e)

6 Final remarks

The optimization results presented are referred only to the thin plate (Kirchhoff) model. This is the only plate model for which a rigorous relaxation of the minimum compliance problem has been put forward.

If one assumes that the so called laminates of finite rank suffice for the relaxation of the minimum compliance problem of Reissner-Hencky plates we have the result of Lipton [19], Lipton and Diaz [20], Diaz et al. [3] according to which the laminates of 3rd rank are sufficient to fill up the design space.

A rigorous three-dimensional approach to plate optimization has been recently presented by Olhoff et al. [8].

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